Stability of a Bose-Einstein condensate revisited for composite bosons

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It is known that elementary bosons condense in a unique state, not so much because this state has the lowest free-particle energy but because it costs a macroscopic amount of energy to put the particles into different states which can then interact through quantum particle exchanges. Since individual exchanges between the two fermions of a composite boson are ignored when composite particles are replaced by elementary bosons, it is of importance to reconsider the exchange-energy argument for the stability of the Bose-Einstein condensate in the case of composite bosons. We do this here in the light of the new many-body theory which allows us to take exactly into account all possible exchanges between the fermionic components of the composite bosons. We confirm that the condensate of composite bosons occupies a single state, this state being moreover *pure*: a coherent superposition of states close in energy is shown to be less favorable for both elementary and composite bosons.

I. INTRODUCTION

Bose-Einstein condensation (BEC) is of high current interest due to recent observations of condensates made of atoms,[1] molecules,[2–5] and polaritons,[6,7] which result from the strong coupling of a photon and an exciton.[8] The experimental observation of a pure exciton condensate,[9,10] however, remains a challenge. It has been recently argued[11] that the condensate in this system should appear in a dark state since this is the lowest-energy state due to the weak valence-conduction repulsive processes which do not exist for excitons with spin $S=(+2,-2)$. Consequently, such a dark condensate cannot be directly seen by optical emission investigated up to now,[12] though it should be possible to deduce its presence indirectly. In addition, excitons are excited states; so that in order to reach condensation, excitons with a lifetime long compared to their thermalization time are needed. This has motivated the development of coupled quantum well structures[13] with electrons well separated from holes to increase their recombination time.

The standard formulation of BEC considers a set of free elementary bosons. For particles with a center-of-mass momentum, a special role is then played by the zero-momentum $(\vec{k}=0)$ state. However, since the energy spectrum of such particles confined in a large volume is essentially continuous, it is physically hard to accept that this lowest-energy state is favored over all the other nearby states just on the account of an infinitesimally small kinetic-energy difference.

Actually, the essential characteristic of a Bose-Einstein condensate, that there is a macroscopic number of particles in a single quantum state, arises not from an argument for an ideal gas but from interactions. This was made in a very straightforward way by Nozières.[14,15] By calculating the energy in the Born approximation of N elementary bosons with repulsive interaction (necessary to avoid a density collapse), he showed that to break up the condensate into two different states, we must pay a macroscopic exchange-energy penalty

which increases as N^2. As Nozières[14] said, "…it is the exchange-interaction energy that makes condensation fragmentation costly. Genuine Bose-Einstein condensation is not an ideal gas effect: it implies interacting particles!"

This nicely shows that exchange between indistinguishable quantum particles plays a fundamental, not peripheral, role in the essential physics of Bose-Einstein condensation. A similar role is actually played in the spin-spin J coupling of ferromagnets, favoring particles to be in the same state: without this coupling, there is no ferromagnetic phase transition. As shown below, even for just $N=2$ bosons, the result is still valid. In other words, there already is an exchange-energy penalty for two bosons to be in different states. As a general rule, we can say that Bose-Einstein condensation occurs when the exchange energy for two bosons becomes comparable to their thermal energy.

As the conclusion of Nozières[14] was obtained before the development of the composite-boson many-body theory, it was de facto reached in the framework of elementary bosons. Yet, all real condensates consist of composite bosons which are made of an even number of fermions. Due to the Pauli exclusion principle between fermions, additional exchange processes between the fermions which make up the composite bosons must be considered in the overall exchange energy. These effects are expected to be important not only for excitons composed of light-mass electron and hole, but also for atoms, since the electron exchange energy is known to enter the effective interaction between two atoms.

A number of works[16–26] have addressed interaction and condensation of composite bosons ("cobosons" in short) such as excitons. One approach is to stay completely in the fermion picture. Treating the correlated pairs in this approach can be cumbersome and can require heavy numerical methods. Another popular approach is the method of bosonization, in which the fermion pairs are treated as pure bosons but with an altered interaction taking into account the underlying Fermi statistics. As discussed in a series of recent papers presenting a new approach to composite-boson

theory,[27,28] the bosonization method neglects certain exchange processes which are important even in low-order perturbation theory. Neglecting these can sometimes have dramatic consequences, for example, neglecting dominant terms in semiconductor optical nonlinearities.[29,30] In the present paper we will use the new composite-boson theory, which has a convenient diagrammatic method which lends itself to analytical results. We reconsider the overall stability of Bose-Einstein condensation in the case of composite bosons using the framework of this new many-body theory since exchange, which is crucial for the condensate stability, is not properly treated when the fermionic components of the particles are forgotten. Our aim in this paper is to show that, indeed, the exchange-energy stability argument still applies in the case of composite bosons, using the new tools that this composite-boson many-body theory now offers.

This many-body theory shows that two composite bosons interact through two conceptually different scatterings:[27] energylike "interaction scatterings" for fermion interactions in the absence of fermion exchange, and dimensionless "Pauli scatterings" for fermion exchanges in the absence of fermion interaction. These Pauli scatterings, by construction ignored when the composite particles are "bosonized," turn out to be crucial in the many-body physics of composite bosons: they, in particular, control all semiconductor optical nonlinearities induced by unabsorbed photons. From these 2×2 Pauli scatterings, we can construct any possible fermion exchange which exists between N composite bosons, which is necessary since the Pauli exclusion principle from which these exchanges originate is N body by essence. These N-body exchanges are nicely visualized through new diagrams, called "Shiva diagrams,"[31] which allow one not only to see the subtle many-body physics taking place between these tricky objects but also to calculate it readily.

This new formalism also shows that it is impossible to write an effective Hamiltonian for bosonized excitons which produces the correct scattering rates and the correct lifetime of N exciton states,[32] even in the extreme dilute limit of $N = 2$. A way to grasp the difficulty is to note that by mapping composite bosons into an elementary-boson subspace, we strongly reduce the degrees of freedom of the problem. This mathematically shows up through a change from $1/N!$ to $(1/N!)^2$ in the prefactor of the closure relation for N elementary or composite bosons,[33] making all sum rules irretrievably different.

In this paper, we consider the possibility of having not only a condensate made of two different states but also a condensate made of a coherent superposition of states close in energy, for both elementary and composite bosons. Thus, we are going to consider the three states

$$|\phi_0\rangle = B_o^{\dagger N}|0\rangle, \qquad (1.1)$$

$$|\phi_{12}\rangle = B_{o_1}^{\dagger N_1} B_{o_2}^{\dagger N_2}|0\rangle, \qquad (1.2)$$

$$|\phi\rangle = (a' B_{o'}^{\dagger} + a'' B_{o''}^{\dagger})^N|0\rangle, \qquad (1.3)$$

where the $B_i^{\dagger}$ operators are creation operators for composite bosons. Moreover, we will compare the results with the ones

obtained with elementary-boson operators, written with a bar as $\bar{B}_i^{\dagger}$. These operators have the standard commutation relation $[\bar{B}_m, \bar{B}_i^{\dagger}] = \delta_{mi}$. For composite bosons, the $B_i^{\dagger}$ operators obey the relation $[B_m, B_i^{\dagger}] = \delta_{mi} - D_{mi}$, where the "deviation-from-boson" operator D_{mi} will be discussed in more detail below. The third state, $|\phi\rangle$, defined in Eq. (1.3) is a coherent superposition of composite bosons in two different eigenstates. The possibility of having such a coherent superposition, not considered by Nozières[14] but considered by Leggett,[15] will also be addressed here in the case of elementary bosons. In the spirit of the Nozières and Leggett arguments, we consider only the lowest order of perturbation theory, calculating the mean-field energy of the different states. This is certainly valid in the low-density limit, although as discussed by Leggett,[15] a rigorous demonstration of the stability of a condensate in the general case of interacting bosons at all densities does not yet exist. We confirm that the mean value of the Hamiltonian is minimum for $|\bar{\phi}_0\rangle$ and $|\phi_0\rangle$, i.e., for a condensate which is pure and not fragmented, whatever the repulsive scatterings between particles are and whether or not the bosons are elementary or composite.

As the calculations for a large number of composite bosons are heavy and quite technical, especially for readers who are not yet familiar with this new many-body theory for composite bosons, we first perform the calculations for $N = 2$. Most of the important physics is usually seen just from examining two interacting particles. A nice rule of thumb—which will be explicitly confirmed—allows us to obtain the results for N by replacing $(a_B/L)^d$ in the results for $N = 2$ with

$$\eta = N(a_B/L)^d, \qquad (1.4)$$

where a_B is the composite-boson wave-function extension, L is the sample size, and d is the spatial dimension. This rule of thumb is less obvious for the fragmented state $|\phi_{12}\rangle$ because we then have two boson populations N_1 and N_2. However, we physically expect to have the exchange terms between the composite bosons in states 1 and 2 to appear with a prefactor $N_1 N_2$, while the ones involving exchange only in state 1 or only in state 2 should appear with prefactors $N_1(N_1 - 1)/2$ and $N_2(N_2 - 1)/2$ for the number of ways to choose two particles among the N_1 or N_2 populations.

As with other many-body effects between composite bosons, the tricky part of the calculation always is to determine the density expansion of the relevant scalar products for N composite-boson states. While it is always possible to get them through a brute-force algebra making use of the commutations[27] on which the composite-boson many-body theory is based, these expansions are nicely performed by using Shiva diagrams[31] which visualize fermion exchanges in a transparent way: the density expansions are naturally associated with diagrams with an increasing number of composite-boson lines. In the case of state $|\phi_{12}\rangle$, however, we must work a little harder because we now have two large numbers which are relevant, N_1 and N_2. In order to determine the extra energy due to the fracturing of the condensate into two different states, we must carefully distinguish between the exchanges which are internal to the N_1 population, those

which are internal to the N_2 population, and those which occur between these two populations. Only the last contribute to the exchange penalty for fragmenting the condensate into two different states.

The paper is organized as follows. In Secs. II and III, we review the formalism for many-body effects with elementary and composite bosons, and settle the notations. An important part of this discussion is the normalization of N-particle states, which is trivial for elementary bosons but far from trivial for composite bosons. In particular, we present results for the normalization of a state made of two large numbers of different cobosons.

In Sec. IV, we calculate the mean value of an effective Hamiltonian describing interacting elementary bosons, in the case where there are *two* elementary bosons in each of the three configurations $|\bar{\phi}_0\rangle$, $|\bar{\phi}_{12}\rangle$, and $|\bar{\phi}\rangle$, while in Sec. V we perform the same calculations for two *composite* bosons, using the exact Hamiltonian for interacting fermions. This in particular, allows us to identify the proper effective scatterings one has to take for diagonal processes between bosonized particles in terms of the interaction and Pauli scatterings appearing in the composite-boson many-body theory.

In Sec. VI, we calculate the effective Hamiltonian mean value for N elementary bosons in the three states of interest, namely, pure, fragmented, and coherent, while in Sec. VII, we address the same problem for N composite bosons. In Sec. VIII, we discuss the results and conclude. In the Appendix, we briefly reproduce the original argument of Nozières,[14] for completeness and also because the spirit of the present paper is the same: by calculating the Hamiltonian mean values in the states $|\bar{\phi}_0\rangle$, $|\bar{\phi}_{12}\rangle$, and $|\bar{\phi}\rangle$, we by construction study the effect of interactions between bosons in the Born approximation, i.e., to first order in the interactions. The states $\bar{B}_0^{\dagger N}|0\rangle$ and $B_0^{\dagger N}|0\rangle$ are the system ground states in zero order for the elementary and composite bosons, respectively. For those states, the mean value of the Hamiltonian reduces to NE_0 in the low-density limit. By drawing our conclusion about the nonfragmentation of the condensate from the sign of the term linear in density, obtained within the Born approximation, we of course implicitly assume that higher-order terms in the interaction will not modify the overall sign, which is likely in view of our past knowledge of many-body effects. This approach assumes zero temperature and low density. Actually, low density is implicitly assumed when considering excitons, which can convert to electron-hole plasma as density increases.[34]

II. FORMALISM FOR ELEMENTARY BOSONS

In this section, we briefly recall the formalism for elementary-boson many-body calculations. The commutation relation for these bosons is

$$[\bar{B}_m, \bar{B}_i] = 0, \quad [\bar{B}_m, \bar{B}_i^{\dagger}] = \delta_{mi}. \tag{2.1}$$

The index i usually stands for a center-of-mass momentum $\vec{Q}_i$ and a relative motion index ν_i. For elementary bosons, it is possible to split the system Hamiltonian as $\bar{H} = \bar{H}_0 + \bar{V}$,
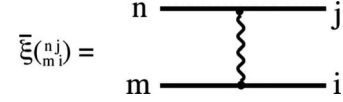

FIG. 1. Scattering $\bar{\xi}\binom{n\ j}{m\ i}$, as defined in Eq. (2.2), between two elementary bosons.

where the one-body part is $\bar{H}_0 = \Sigma E_i \bar{B}_i^{\dagger} \bar{B}_i$, while the interaction can be written as

$$\bar{V} = \frac{1}{2} \sum \bar{\xi}\binom{n\ j}{m\ i} \bar{B}_m^{\dagger} \bar{B}_n^{\dagger} \bar{B}_j \bar{B}_i. \tag{2.2}$$

The energylike prefactor $\bar{\xi}\binom{n\ j}{m\ i}$, which describes the scattering from i to m and j to n (see Fig. 1) must be such that $\bar{\xi}\binom{n\ j}{m\ i}^* = \bar{\xi}\binom{j\ n}{i\ m}$ to ensure hermiticity. Momentum conservation requires that this scattering differs from zero only for $\vec{Q}_m + \vec{Q}_n = \vec{Q}_i + \vec{Q}_j$. The argument of Nozières[14] was made for a structureless scattering, i.e., for $\bar{\xi}\binom{n\ j}{m\ i} = V_0 \delta_{\vec{Q}_m + \vec{Q}_n, \vec{Q}_i + \vec{Q}_j}$. By noting that $\bar{B}_m^{\dagger} \bar{B}_n^{\dagger} = \bar{B}_n^{\dagger} \bar{B}_m^{\dagger}$, it is possible to rewrite this potential $\bar{V}$ in a fully symmetrical form,

$$\bar{V} = \frac{1}{2} \sum \bar{\xi}_{mn;ij} \bar{B}_m^{\dagger} \bar{B}_n^{\dagger} \bar{B}_j \bar{B}_i, \tag{2.3}$$

where $\bar{\xi}_{mn;ij}$, equal to $[\bar{\xi}\binom{n\ j}{m\ i} + \bar{\xi}\binom{m\ j}{n\ i}]/2$, is now such that $\bar{\xi}_{mn;ij} = \bar{\xi}_{mn;ji} = \bar{\xi}_{nm;ij} = \bar{\xi}_{ij;mn}^*$.

In order to perform many-body calculations with elementary bosons in a convenient way, let us note that

$$[\bar{B}_m, \bar{B}_o^{\dagger N}] = [\bar{B}_m, \bar{B}_o^{\dagger}]\bar{B}_o^{\dagger N-1} + \bar{B}_o^{\dagger}[\bar{B}_m, \bar{B}_o^{\dagger}]\bar{B}_o^{\dagger N-2} + \cdots$$
$$= N\delta_{mo}\bar{B}_o^{\dagger N-1}. \tag{2.4}$$

This gives for the state $|\bar{\phi}_o\rangle$ defined in Eq. (1.1) the following:

$$\bar{B}_j \bar{B}_i |\bar{\phi}_o\rangle = N(N-1)\delta_{io}\delta_{jo}\bar{B}_o^{\dagger N-2}|0\rangle. \tag{2.5}$$

If instead of a single state, we now consider a coherent superposition of two states, $\bar{B}^{\dagger} = a'\bar{B}_{o'}^{\dagger} + a''\bar{B}_{o''}^{\dagger}$, Eq. (2.4) implies

$$[\bar{B}_i, \bar{B}^{\dagger N}] = N\delta_i \bar{B}^{\dagger N-1}, \tag{2.6}$$

with δ_i such that $[\bar{B}_i, \bar{B}^{\dagger}] = \delta_i$, i.e.,

$$\delta_i = a'\delta_{io'} + a''\delta_{io''}. \tag{2.7}$$

III. FORMALISM FOR COMPOSITE BOSONS

A. Elementary scattering

We now consider composite bosons made of one fermion α and one fermion β, and we briefly review the formalism presented in Refs. 27 and 28. The Hamiltonian of these fermions reads

$$H = H_{\alpha} + H_{\beta} + V_{\alpha\alpha} + V_{\beta\beta} + V_{\alpha\beta}. \tag{3.1}$$

For excitons or hydrogen atoms, $V_{\alpha\alpha}$, $V_{\beta\beta}$, and $V_{\alpha\beta}$ are Coulomb potentials, while for the so-called cold Fermi gases,

$V_{\alpha\alpha} \approx V_{\beta\beta} \approx 0$, while $V_{\alpha\beta}$ is short range. These interaction terms can be formally written as

$$V_{\alpha\beta} = \sum \mathcal{V}\begin{pmatrix} \vec{k}'_\beta & \vec{k}'_\beta \\ \vec{k}'_\alpha & \vec{k}_\alpha \end{pmatrix} a^\dagger_{\vec{k}'_\alpha} b^\dagger_{\vec{k}'_\beta} b_{\vec{k}_\beta} a_{\vec{k}_\alpha} \qquad (3.2)$$

and similarly for $V_{\alpha\alpha}$ and $V_{\beta\beta}$ with a 1/2 prefactor. $a^\dagger_{\vec{k}_\alpha}$ and $b^\dagger_{\vec{k}_\beta}$ are, for simplicity, chosen to be the creation operators for one fermion α or β in an eigenstate of the system Hamiltonian, namely, $|\vec{k}_\alpha\rangle = a^\dagger_{\vec{k}_\alpha}|0\rangle$ and $|\vec{k}_\beta\rangle = b^\dagger_{\vec{k}_\beta}|0\rangle$, with $(H - \varepsilon_{\vec{k}_\gamma})|\vec{k}_\gamma\rangle = 0$ for $\gamma = \alpha$ or β.

The creation operators of the one-pair eigenstates $|i\rangle = B^\dagger_i|0\rangle$ of the system Hamiltonian, with $(H - E_i)|i\rangle = 0$, can be written in terms of these free-fermion operators as

$$B^\dagger_i = \sum a^\dagger_{\vec{k}_\alpha} b^\dagger_{\vec{k}_\beta} \langle \vec{k}_\beta \vec{k}_\alpha | i \rangle. \qquad (3.3)$$

In the same way, the fermion pair-creation operators can be written in terms of the $B^\dagger_i$'s as

$$a^\dagger_{\vec{k}_\alpha} b^\dagger_{\vec{k}_\beta} = \sum B^\dagger_i \langle i | \vec{k}_\alpha \vec{k}_\beta \rangle. \qquad (3.4)$$

$\langle \vec{k}_\beta \vec{k}_\alpha | i \rangle$ is the wave function of the eigenstate i in momentum space if (H_α, H_β) only contain free-energy contributions. These two equations, (3.3) and (3.4), are quite fundamental in the composite-boson many-body theory as they allow us to "open" the composite bosons into their fermionic components to let the fermions interact through their exact interaction potentials and then to "close" these fermions back into composite bosons after their interactions.

The $B^\dagger_i$'s are composite-boson operators. Indeed, their behavior is like bosons since

$$[B_m, B_i] = 0. \qquad (3.5)$$

However, since these operators are such that

$$[B_m, B^\dagger_i] = \delta_{mi} - D_{mi}, \qquad (3.6)$$

they differ from elementary bosons, the deviation from elementary-boson statistics appearing in the operator D_{mi}. This operator is such that $D_{mi}|0\rangle = 0$, as can be seen by taking the above equation acting on the vacuum state $|0\rangle$. The Pauli scatterings, which describe fermion exchanges between two cobosons in the absence of fermion interaction, are obtained from these deviation-from-boson operators D_{mi}, through the commutation relation

$$[D_{mi}, B^\dagger_j] = \sum \left\{ \lambda\begin{pmatrix} n & j \\ m & i \end{pmatrix} + (m \leftrightarrow n) \right\} B^\dagger_n, \qquad (3.7)$$

where the second term means the same as the first but with m and n interchanged. $\lambda\binom{n\ j}{m\ i}$, which is a dimensionless factor, describes the fermion exchange between cobosons (i,j) shown in Fig. 2(a).

To get interaction scatterings which are energylike quantities, we introduce the "creation potential" $V^\dagger_i$, which arises from the commutation relation[27,35]
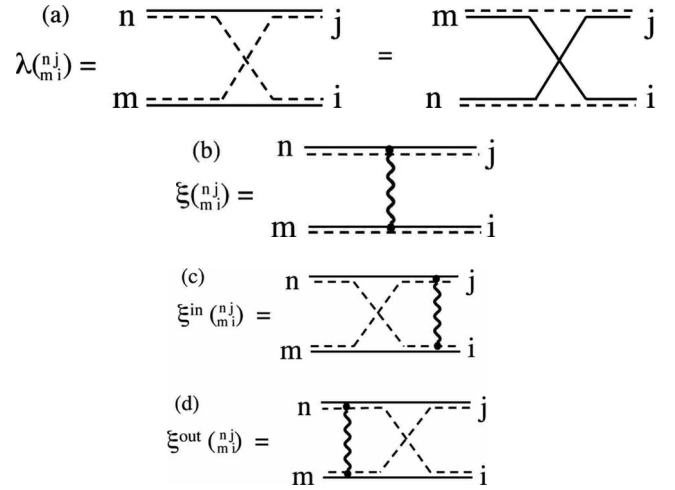

FIG. 2. (a) Pauli scattering $\lambda\binom{n\ j}{m\ i}$, as defined in Eq. (3.7), for fermion exchange between two composite bosons in the absence of fermion interaction. Note that this scattering can be seen as an exchange of fermions β or an exchange of fermions α with the indices (m,n) permutated. (b) Interaction scattering $\xi\binom{n\ j}{m\ i}$, as defined in Eq. (3.9), for fermion interaction between two composite bosons in the absence of fermion exchange. (c) Exchange-interaction scattering $\xi^{in}\binom{n\ j}{m\ i}$, as defined in Eq. (3.10), when the fermion exchange takes place *after* the interaction, so that these interactions are between the in cobosons (i,j). (d) Exchange-interaction scattering $\xi^{out}\binom{n\ j}{m\ i}$, as defined in Eq. (3.10), when the fermion exchange occurs *before* the interaction, so that these interactions are between the out cobosons (m,n).

$$[H, B^\dagger_i] = E_i B^\dagger_i + V^\dagger_i, \qquad (3.8)$$

so that $V^\dagger_i|0\rangle = 0$ for $B^\dagger_i|0\rangle$ being the H eigenstate with energy E_i. The interaction scatterings $\xi\binom{n\ j}{m\ i}$ then appear through the commutation relation[27]

$$[V^\dagger_i, B^\dagger_j] = \sum \xi\begin{pmatrix} n & j \\ m & i \end{pmatrix} B^\dagger_m B^\dagger_n. \qquad (3.9)$$

The prefactor $\xi\binom{n\ j}{m\ i}$, shown in Fig. 2(b), describes the direct scattering of the "incoming" cobosons (i,j) resulting from the interactions of their fermions, the "outgoing" cobosons (m,n) being made with the same fermion pairs. From $\xi\binom{n\ j}{m\ i}$, it is possible to construct the "in" and "out" exchange-interaction scatterings shown in Figs. 2(c) and 2(d) through a sequence of direct-interaction scattering and dimensionless Pauli scattering, according to

$$\xi^{in}\begin{pmatrix} n & j \\ m & i \end{pmatrix} = \sum \lambda\begin{pmatrix} n & q \\ m & p \end{pmatrix} \xi\begin{pmatrix} q & j \\ p & i \end{pmatrix},$$

$$\xi^{out}\begin{pmatrix} n & j \\ m & i \end{pmatrix} = \sum \xi\begin{pmatrix} n & q \\ m & p \end{pmatrix} \lambda\begin{pmatrix} q & j \\ p & i \end{pmatrix}. \qquad (3.10)$$

In these exchange-interaction scatterings, the "in" and "out" cobosons are made with different pairs, the fermion interactions taking place between the "in" cobosons in $\xi^{in}\binom{n\ j}{m\ i}$ and between the "out" cobosons in $\xi^{out}\binom{n\ j}{m\ i}$. These exchange-

interaction scatterings are linked to the Pauli scattering through[28]

$$\xi^{\text{in}}\begin{pmatrix} n & j \\ m & i \end{pmatrix} - \xi^{\text{out}}\begin{pmatrix} n & j \\ m & i \end{pmatrix} = (E_m + E_n - E_i - E_j)\lambda\begin{pmatrix} n & j \\ m & i \end{pmatrix},$$

(3.11)

so that they are equal for energy-conserving processes. Since both interaction processes and fermion exchanges conserve momenta, all these scatterings differ from zero only for $\vec{Q}_m + \vec{Q}_n = \vec{Q}_i + \vec{Q}_j$.

In the following, we will see that the physically relevant combinations of energylike scatterings appear to be

$$\hat{\xi}\begin{pmatrix} n & j \\ m & i \end{pmatrix} = \xi\begin{pmatrix} n & j \\ m & i \end{pmatrix} - \xi^{\text{in}}\begin{pmatrix} n & j \\ m & i \end{pmatrix},$$

(3.12)

which we are going to symmetrize as

$$\hat{\xi}_{mn;ij} = \frac{1}{2}\hat{\xi}\begin{pmatrix} n & j \\ m & i \end{pmatrix} + (m \leftrightarrow n),$$

(3.13)

in order to have $\hat{\xi}_{mn;ij} = \hat{\xi}_{nm;ij} = \hat{\xi}_{mn;ji}$. In the same way, we introduce the symmetrized Pauli scatterings $\lambda_{mn;ij}$, equal to $[\lambda\begin{pmatrix} n & j \\ m & i \end{pmatrix} + (m \leftrightarrow n)]/2$.

B. Many-body effects

To derive many-body effects with identical cobosons, it is convenient to iterate the four commutation relations (3.5)–(3.9) on which the composite-boson many-body theory is based. This leads to[27,28]

$$[B_m, B_o^{\dagger N}] = NB_o^{\dagger N-1}(\delta_{mo} - D_{mo})$$
$$- N(N-1)B_o^{\dagger N-2}\sum \lambda\begin{pmatrix} n & o \\ m & o \end{pmatrix}B_n^{\dagger}, \quad (3.14)$$

$$[D_{mi}, B_o^{\dagger N}] = NB_o^{\dagger N-1}\sum \left\{ \lambda\begin{pmatrix} n & i \\ m & o \end{pmatrix} + (m \leftrightarrow n) \right\} B_n^{\dagger}$$

(3.15)

for many-body effects dealing with fermion exchanges and

$$[H, B_o^{\dagger N}] = NB_o^{\dagger N-1}(E_o B_o^{\dagger} + V_o^{\dagger})$$
$$+ \frac{N(N-1)}{2}B_o^{\dagger N-2}\sum \xi\begin{pmatrix} n & o \\ m & o \end{pmatrix}B_m^{\dagger}B_n^{\dagger},$$

(3.16)

$$[V_i^{\dagger}, B_o^{\dagger N}] = NB_o^{\dagger N-1}\sum \xi\begin{pmatrix} n & o \\ m & i \end{pmatrix}B_m^{\dagger}B_n^{\dagger}$$

(3.17)

for many-body effects dealing with fermion interactions. The N prefactors in front of the λ and ξ terms come from the number of ways among N to choose the cobosons in state o involved in these scatterings. This gives N when only one coboson in state o is involved and $N(N-1)/2$ when two cobosons in state o are involved, the additional 2 in Eq. (3.14) coming from the fact that these cobosons can exchange ei-

ther their fermion β or their fermion α, the $(\alpha \leftrightarrow \beta)$ exchange being identical to an $(i \leftrightarrow j)$ exchange, as seen in Fig. 2(a).

In typical problems dealing with N cobosons, most of them are in the same state. Scalar products of such N coboson states are easy to expand[31] in terms of

$$\langle 0|B_o^N B_o^{\dagger N}|0\rangle = N!\, F_N.$$

(3.18)

The normalization factor F_N reduces to 1 for the case of elementary bosons, but it is not a number on the order of 1 for cobosons due to the Pauli exclusion principle between the fermionic components of the particles. Indeed, from the recursion relation obeyed by F_N, it has been shown[28] that for large samples, F_N is exponentially small ($F_N \sim e^{-N\eta}$), where η is the dimensionless parameter associated with density defined by Eq. (1.4). However, in calculations of physical quantities, F_N always appears in ratios F_{N-n}/F_N which, for n small, is equal to 1 within corrections on the order of η,

$$\frac{F_{N-1}}{F_N} = 1 + \mathcal{O}(\eta).$$

(3.19)

C. Coherent superposition of cobosons

Besides many-body states with a given number of particles in a specific eigenstate, we wish also to consider coherent superpositions of cobosons, i.e., states of the form

$$B^{\dagger} = a'B_{o'}^{\dagger} + a''B_{o''}^{\dagger}.$$

(3.20)

While calculations for $N=2$ are easy to perform by simply expanding $(a'B_{o'}^{\dagger} + a''B_{o''}^{\dagger})^2$, calculations for large N are more tricky. For those, we are forced to keep the coherent state $B^{\dagger}$ as an entity, and to produce equations similar to Eqs. (3.14)–(3.17) with $B_o^{\dagger}$ replaced by $B^{\dagger}$.

1. Fermion exchanges

To get the fermion exchanges in a convenient way, let us first introduce the operator D_m through the commutator $[B_m, B^{\dagger}] = \delta_m - D_m$, where δ_m is defined in Eq. (2.7). D_m reads in terms of the deviation-from-boson operators as $D_m = a'D_{mo'} + a''D_{mo''}$. From it, we then construct the λ_{mn}'s through the commutator $[D_m, B^{\dagger}] = 2\sum \lambda_{mn}B_n^{\dagger}$. They read in terms of the Pauli scatterings as

$$\lambda_{mn} = a'^2\lambda\begin{pmatrix} n & o' \\ m & o' \end{pmatrix} + a''^2\lambda\begin{pmatrix} n & o'' \\ m & o' \end{pmatrix}$$
$$+ a'a''\left[\lambda\begin{pmatrix} n & o'' \\ m & o' \end{pmatrix} + \lambda\begin{pmatrix} n & o' \\ m & o'' \end{pmatrix} \right].$$

(3.21)

Iteration of these two commutators allows us to write $[D_m, B^{\dagger N}]$ and $[B_m, B^{\dagger N}]$ in terms of this λ_{mn}. We find

$$[D_m, B^{\dagger N}] = 2NB^{\dagger N-1}\sum \lambda_{mn}B_n^{\dagger},$$

(3.22)

$$[B_m, B^{\dagger N}] = NB^{\dagger N-1}(\delta_m - D_m) - N(N-1)B^{\dagger N-2}\sum \lambda_{mn}B_n^{\dagger}.$$

(3.23)

From these two commutators, it is also possible to show that $[B, B^{\dagger}] = 1 - D$, while $[D, B^{\dagger}] = 2\sum \lambda_n B_n^{\dagger}$, in which the prefac-

tor λ_n reads in terms of the Pauli scatterings as

$$\lambda_n = a'|a'|^2 \lambda \begin{pmatrix} n & o' \\ o' & o' \end{pmatrix} + a''|a''|^2 \lambda \begin{pmatrix} n & o'' \\ o'' & o'' \end{pmatrix}$$
$$+ a''|a'|^2 \left\{ \lambda \begin{pmatrix} n & o'' \\ o' & o' \end{pmatrix} + \lambda \begin{pmatrix} n & o' \\ o' & o'' \end{pmatrix} \right\}$$
$$+ a'|a''|^2 \left\{ \lambda \begin{pmatrix} n & o' \\ o'' & o'' \end{pmatrix} + \lambda \begin{pmatrix} n & o'' \\ o'' & o' \end{pmatrix} \right\}. \quad (3.24)$$

Similar commutators for N, obtained by iteration, read in terms of this λ_n as

$$[D, B^{\dagger N}] = 2N B^{\dagger N-1} \sum \lambda_n B_n^{\dagger}, \quad (3.25)$$

$$[B, B^{\dagger N}] = N B^{\dagger N-1}(1-D) - N(N-1) B^{\dagger N-2} \sum \lambda_n B_n^{\dagger}. \quad (3.26)$$

These two commutation relations, along with Eqs. (3.22) and (3.23), are the equivalent of Eqs. (3.14) and (3.15) for coherent superpositions of cobosons.

2. Fermion interactions

To perform calculations for N of these coherent superpositions in a convenient way, we also need similar commutators for the interaction part. They are obtained through $[H, B^{\dagger}] = E_o \widetilde{B}^{\dagger} + V^{\dagger}$, in which we have set $V^{\dagger} = a' V_{o'}^{\dagger} + a'' B_{o''}^{\dagger}$, while

$$\widetilde{B}^{\dagger} = (a' E_{o'} B_{o'}^{\dagger} + a'' E_{o''} B_{o''}^{\dagger})/E_o, \quad (3.27)$$

so that $\widetilde{B}^{\dagger}$ reduces to $B^{\dagger}$ for $E_{o'} \approx E_{o''} \approx E_o$. We are then led to define ξ_{mn} through $[V^{\dagger}, B^{\dagger}] = \Sigma \xi_{mn} B_m^{\dagger} B_n^{\dagger}$. This scalar is found to be

$$\xi_{mn} = a'^2 \xi \begin{pmatrix} n & o' \\ m & o' \end{pmatrix} + a''^2 \xi \begin{pmatrix} n & o'' \\ m & o'' \end{pmatrix}$$
$$+ a'a'' \left\{ \xi \begin{pmatrix} n & o'' \\ m & o' \end{pmatrix} + \xi \begin{pmatrix} n & o' \\ m & o'' \end{pmatrix} \right\}, \quad (3.28)$$

so that the iteration of these commutators leads to

$$[V^{\dagger}, B^{\dagger N}] = N B^{\dagger N-1} \sum \xi_{mn} B_m^{\dagger} B_n^{\dagger}, \quad (3.29)$$

$$[H, B^{\dagger N}] = N B^{\dagger N-1}(E_o \widetilde{B}^{\dagger} + V^{\dagger}) + \frac{N(N-1)}{2} B^{\dagger N-2} \sum \xi_{mn} B_m^{\dagger} B_n^{\dagger}. \quad (3.30)$$

3. Normalization factors

As for single composite bosons, the norm of the state made of N identical coherent superpositions of cobosons is going to play a key role in the many-body physics of these systems. This leads us to introduce, just as we defined F_N for single cobosons, the normalization factor G_N, determined by

$$\langle 0| B^N B^{\dagger N} |0\rangle = N! \, G_N. \quad (3.31)$$

G_N is expected to be exponentially small due to the many exchanges which take place between the fermions of the N coherent bosons. As for F_N, we reach G_N through the recursion relation it obeys. From Eq. (3.26) and the fact that $D|0\rangle = 0$, we find

$$\langle 0| B^N B^{\dagger N} |0\rangle = \langle 0| B^{N-1}[B, B^{\dagger N}]|0\rangle$$
$$= N(N-1)! \, G_{N-1} - N(N-1)$$
$$\times \sum \lambda_n \langle 0| B^{N-1} B_n^{\dagger} B^{\dagger N-2}|0\rangle. \quad (3.32)$$

We now use Eq. (3.23) to get $\langle 0| B^{N-1} B_n^{\dagger}$. This leads to

$$G_N = G_{N-1} - (N-1)\lambda^{(1)} G_{N-2} + (N-1)(N-2)\lambda^{(2)} G_{N-3} + \cdots, \quad (3.33)$$

where the precise value of the first-order term $\lambda^{(1)}$ is

$$\lambda^{(1)} = \sum \lambda_n \delta_n^* = |a'|^4 \lambda \begin{pmatrix} o' & o' \\ o' & o' \end{pmatrix} + |a''|^4 \lambda \begin{pmatrix} o'' & o'' \\ o'' & o'' \end{pmatrix}$$
$$+ 2|a'a''|^2 \left\{ \lambda \begin{pmatrix} o'' & o'' \\ o' & o' \end{pmatrix} + \lambda \begin{pmatrix} o'' & o' \\ o' & o'' \end{pmatrix} \right\}. \quad (3.34)$$

Equations (3.33) and (3.34) show that like F_N, the G_N correction to the bare elementary-boson normalization factor $N!$, although not on the order of 1, is such that

$$\frac{G_{N-1}}{G_N} \simeq 1 + N\lambda^{(1)} = 1 + \mathcal{O}(\eta). \quad (3.35)$$

D. Mixture of cobosons

In this paper, we are also going to consider a mixture of cobosons of the form $B_{o_1}^{\dagger N_1} B_{o_2}^{\dagger N_2}|0\rangle$. Like F_N for many-body effects between N identical cobosons, G_{N_1, N_2}, defined by

$$\langle 0| B_{o_2}^{N_2} B_{o_1}^{N_1} B_{o_1}^{\dagger N_1} B_{o_2}^{\dagger N_2}|0\rangle = N_1! \, N_2! \, G_{N_1, N_2}, \quad (3.36)$$

is a key factor for many-body effects between N_1 cobosons in state o_1 and N_2 cobosons in state o_2. This factor, which would be exactly 1 for elementary bosons, differs from 1 due to the many fermion exchanges which exist not only among the N_1 cobosons in state o_1 and among the N_2 cobosons in state o_2, but also between the cobosons in state o_1 and the cobosons in state o_2. They make G_{N_1, N_2} not on the order of 1 but exponentially small. As with F_N, we can determine G_{N_1, N_2} through the recursion relation it fulfills. To get this recursion relation, we use Eq. (3.14) to rewrite $B_{o_1} B_{o_1}^{\dagger N_1}$. This leads to

$$\langle 0| B_{o_2}^{N_2} B_{o_1}^{N_1} B_{o_1}^{\dagger N_1} B_{o_2}^{\dagger N_2}|0\rangle$$
$$= \langle 0| B_{o_2}^{N_2} B_{o_1}^{N_1-1} \left\{ B_{o_1}^{\dagger N_1} B_{o_1} + N_1 B_{o_1}^{\dagger N_1-1}(1 - D_{o_1 o_1}) \right.$$
$$\left. - N_1(N_1-1) B_{o_1}^{\dagger N_1-2} \sum \lambda \begin{pmatrix} i & o_1 \\ o_1 & o_1 \end{pmatrix} B_i^{\dagger} \right\} B_{o_2}^{\dagger N_2}|0\rangle. \quad (3.37)$$
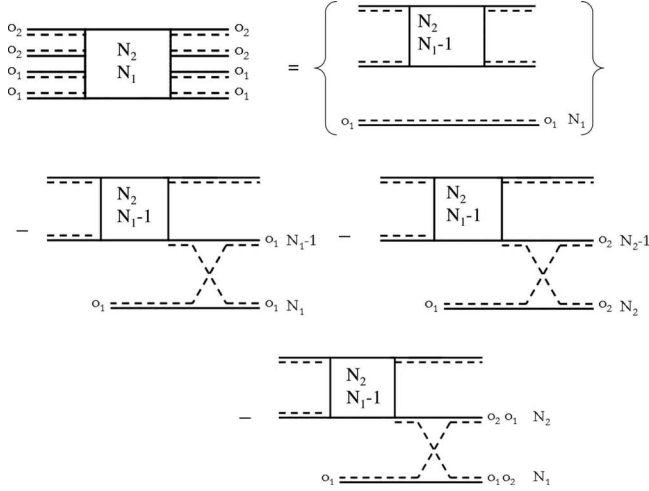
FIG. 3. Shiva diagram expansion for the normalization factor $\langle 0|B_{o_2}^{N_2}B_{o_1}^{N_1}B_{o_1}^{\dagger N_1}B_{o_2}^{\dagger N_2}|0\rangle$ appearing in Eq. (3.38).

The 1 in the second term in the bracket readily gives $N_1[(N_1-1)!N_2!G_{N_1-1,N_2}]$. The first term in the bracket is calculated using Eq. (3.14) for $B_{o_1}B_{o_2}^{\dagger N_2}|0\rangle$, while the third term is calculated using Eq. (3.15) for $D_{o_1o_1}B_{o_2}^{\dagger N_2}|0\rangle$. This allows us to write the norm of $B_{o_1}^{\dagger N_1}B_{o_2}^{\dagger N_2}|0\rangle$ given in Eq. (3.36) as

$$N_2!N_2!G_{N_1,N_2} = N_1[(N_1-1)!N_2!G_{N_1-1,N_2}]$$
$$- N_1(N_1-1)A_{11} - N_2(N_2-1)A_{22}$$
$$- N_1N_2A_{12}, \qquad (3.38)$$

where A_{11}, A_{22}, and A_{12} contain one Pauli scattering explicitly, so that they are related to fermion exchanges. Equation (3.38) is shown in Fig. 3, the factors (N_1,N_2) coming from the number of ways we can choose the cobosons involved in fermion exchanges with the coboson o_1 on the left. The leading term for $N_1(N_1-1)A_{11}$, shown in Fig. 4, reduces to

$$(N_1-1)\lambda\begin{pmatrix} o_1 & o_1 \\ o_1 & o_1 \end{pmatrix}N_1(N_1-1)[(N_1-2)!N_2!G_{N_1-2,N_2}].$$
$$(3.39)$$

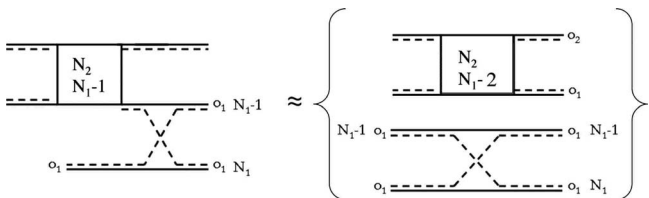Similarly, the leading term for $N_1N_2A_{12}$, shown in Fig. 5, is

FIG. 4. Shiva diagram expansion for the first exchange contribution $N_1(N_1-1)A_{11}$ to $\langle 0|B_{o_2}^{N_2}B_{o_1}^{N_1}B_{o_1}^{\dagger N_1}B_{o_2}^{\dagger N_2}|0\rangle$, as given in Eq. (3.39).
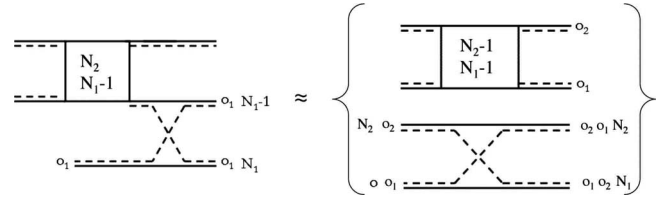

FIG. 5. Same as Fig. 4 for the third exchange contribution $N_1N_2A_{12}$, as given in Eq. (3.40).

$$N_2\left[\lambda\begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix} + \lambda\begin{pmatrix} o_2 & o_1 \\ o_1 & o_2 \end{pmatrix}\right]N_1N_2$$
$$\times[(N_1-1)!(N_2-1)!G_{N_1-1,N_2-1}]. \qquad (3.40)$$

They contain fermion exchanges between two cobosons. On the opposite, the leading term of $N_2(N_2-1)A_{22}$ must have fermion exchange between three cobosons since $\lambda\begin{pmatrix} o_1 & o_2 \\ o_2 & o_2 \end{pmatrix}=0$ for $o_1 \neq o_2$. Therefore, we end with

$$G_{N_1,N_2} = G_{N_1-1,N_2} - (N_1-1)\lambda\begin{pmatrix} o_1 & o_1 \\ o_1 & o_1 \end{pmatrix}G_{N_1-2,N_2}$$
$$- N_2\left[\lambda\begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix} + \lambda\begin{pmatrix} o_2 & o_1 \\ o_1 & o_2 \end{pmatrix}\right]G_{N_1-1,N_2-1} + \cdots.$$
$$(3.41)$$

This shows that, in the same way that $F_{N-1}/F_N \approx 1+\mathcal{O}(\eta)$, we have

$$\frac{G_{N_1-1,N_2}}{G_{N_1,N_2}} = 1 + \mathcal{O}(\eta_1) + \mathcal{O}(\eta_2), \qquad (3.42)$$

so that, even if G_{N_1,N_2} is exponentially small due to the large number of fermion exchanges, the effect of these exchanges on G_{N_1,N_2} ratios is negligible at lowest order in density.

We now have all the tools to tackle many-body effects with a large number of identical cobosons, a large number of coherent cobosons, and a mixture of two large numbers of cobosons. However, since the calculations for large N's are obviously quite technical, in this paper we have chosen to start with $N=2$, as most of the many-body physics can usually be understood from this limit. We are also going to first perform these calculations for elementary bosons, in order to enlighten the differences between elementary and composite particles.

IV. TWO ELEMENTARY BOSONS

A. Single state

Let us start with a state with two identical elementary bosons, $|\bar{\phi}_0\rangle = \bar{B}_o^{\dagger 2}|0\rangle$. Since $\bar{B}_i|\bar{\phi}_0\rangle = 2\delta_{io}\bar{B}_o^{\dagger}|0\rangle$ due to Eq. (2.4), we find that $\langle\bar{\phi}_0|\bar{\phi}_0\rangle = 2$, while

$$\langle\bar{\phi}_0|\bar{H}_0|\bar{\phi}_0\rangle = \sum E_i\{\langle\bar{\phi}_0|\bar{B}_i^{\dagger}\}\{\bar{B}_i|\bar{\phi}_0\rangle\} = 4E_o, \qquad (4.1)$$

$$\langle \bar{\phi}_0 | \bar{V} | \bar{\phi}_0 \rangle = \frac{1}{2} \sum \bar{\xi} \begin{pmatrix} n & j \\ m & i \end{pmatrix} \langle \bar{\phi}_0 | \bar{B}_m^\dagger \bar{B}_n^\dagger \bar{B}_j \bar{B}_i | \bar{\phi}_0 \rangle = 2\bar{\xi} \begin{pmatrix} o & o \\ o & o \end{pmatrix}.$$

(4.2)

This readily gives the Hamiltonian mean value in this two-elementary-boson state as

$$\frac{\langle \bar{\phi}_0 | \bar{H} | \bar{\phi}_0 \rangle}{\langle \bar{\phi}_0 | \bar{\phi}_0 \rangle} = 2E_o + \bar{\xi} \begin{pmatrix} o & o \\ o & o \end{pmatrix} = 2E_o + \bar{\xi}_{oo;oo}, \qquad (4.3)$$

where $\bar{\xi}_{mn;ij}$ is defined in terms of $\bar{\xi} \begin{pmatrix} n & i \\ m & j \end{pmatrix}$ as in Eq. (3.13).

B. Fragmented state

We now consider $|\bar{\phi}_{12}\rangle = \bar{B}_{o_1}^\dagger \bar{B}_{o_2}^\dagger |0\rangle$ for elementary bosons in states $o_1 \neq o_2$. From $\bar{B}_i \bar{\phi}12\rangle$, we readily find $\langle \bar{\phi}_{12} | \bar{\phi}_{12} \rangle = 1$, while $\langle \bar{\phi}_{12} | \bar{H}_0 | \bar{\phi}_{12} \rangle = E_{o_1} + E_{o_2}$. In the same way, $\bar{B}_j \bar{B}_i | \bar{\phi}_{12} \rangle = (\delta_{jo_2} \delta_{io_1} + \delta_{jo_1} \delta_{io_2}) |0\rangle$ leads to

$$\langle \bar{\phi}_{12} | \bar{V} | \bar{\phi}_{12} \rangle = \bar{\xi} \begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix} + \bar{\xi} \begin{pmatrix} o_1 & o_2 \\ o_2 & o_1 \end{pmatrix}, \qquad (4.4)$$

so that we end up with

$$\frac{\langle \bar{\phi}_{12} | \bar{H} | \bar{\phi}_{12} \rangle}{\langle \bar{\phi}_{12} | \bar{\phi}_{12} \rangle} = E_{o_1} + E_{o_2} + 2\bar{\xi}_{o_1 o_2; o_1 o_2} \approx 2E_o + 2\bar{\xi}_{oo;oo}$$

(4.5)

for $o_1 \approx o_2 \approx o$.

C. Coherent superposition

The third state of interest is the coherent superposition of states, $|\bar{\phi}\rangle = \bar{B}^{\dagger 2} |0\rangle$, with $\bar{B}^\dagger$ defined as in Eq. (3.20), with $B_{o'}^\dagger$ replaced by $\bar{B}_{o'}^\dagger$. The prefactors (a', a'') are chosen such that $|a'|^2 + |a''|^2 = 1$ in order for the state to be normalized. We also impose that the (o', o'') states differ at least through their center-of-mass momenta $(\vec{Q}_{o'} \neq \vec{Q}_{o''})$ in order for scatterings such as $\bar{\xi} \begin{pmatrix} o'' & o' \\ o' & o' \end{pmatrix}$ to cancel due to momentum conservation. From Eq. (2.6) taken for $N=2$, we readily find $\langle \bar{\phi} | \bar{\phi} \rangle = 2$, while $\langle \bar{\phi} | \bar{H}_0 | \bar{\phi} \rangle = 4(|a'|^2 E_{o'} + |a''|^2 E_{o''})$. In the same way, from $\bar{B}_j \bar{B}_i | \bar{\phi} \rangle = 2(a' \delta_{io'} + a'' \delta_{io''})(a' \delta_{jo'} + a'' \delta_{jo''}) |0\rangle$ and the fact that the nonzero $\bar{\xi}$'s reduce for $\vec{Q}_{o'} \neq \vec{Q}_{o''}$ to the diagonal terms $\bar{\xi} \begin{pmatrix} i & i \\ j & j \end{pmatrix}$, with $(i,j) = (o'$ or $o'')$, or to the cross term $\bar{\xi} \begin{pmatrix} o' & o'' \\ o'' & o' \end{pmatrix}$, we find that

$$\langle \bar{\phi} | \bar{V} | \bar{\phi} \rangle = 2|a'|^4 \bar{\xi} \begin{pmatrix} o' & o' \\ o' & o' \end{pmatrix} + 2|a''|^4 \xi \begin{pmatrix} o'' & o'' \\ o'' & o'' \end{pmatrix}$$
$$+ 4|a'a''|^2 \left[\xi \begin{pmatrix} o'' & o'' \\ o' & o' \end{pmatrix} + \xi \begin{pmatrix} o' & o'' \\ o'' & o' \end{pmatrix} \right]. \quad (4.6)$$

Consequently, the Hamiltonian mean value for two elementary bosons in a coherent state reads as

$$\frac{\langle \bar{\phi} | \bar{H} | \bar{\phi} \rangle}{\langle \bar{\phi} | \bar{\phi} \rangle} = 2(|a'|^2 E_{o'} + |a''|^2 E_{o''}) + |a'|^4 \bar{\xi}_{o'o';o'o'}$$

$$+ |a''|^4 \bar{\xi}_{o''o'';o''o''} + 4|a'a''|^2 \bar{\xi}_{o'o'';o'o''}$$

$$\approx 2E_o + (1 + 2|a'a''|^2) \bar{\xi}_{oo;oo} \qquad (4.7)$$

for $o' \approx o'' \approx o$.

From Eqs. (4.3), (4.5), and (4.7), we thus see that, since the interaction scattering $\xi_{oo;oo}$ must be positive to avoid a density collapse, the energy of two elementary bosons is minimum when the particles are in a single eigenstate, and not a mixture or a coherent superposition of two eigenstates.

V. TWO COMPOSITE BOSONS

The calculations for composite bosons will need to explicitly treat the exchange between the constituent fermions, through the deviation-from-boson operator D_{mi} and the Pauli scattering $\lambda \begin{pmatrix} n & j \\ m & i \end{pmatrix}$.

A. Single state

For $|\phi_0\rangle = B_o^{\dagger 2} |0\rangle$, we now have, due to Eq. (3.14),

$$B_m |\phi_0\rangle = 2\delta_{mo} B_o^\dagger |0\rangle - 2 \sum \lambda \begin{pmatrix} n & o \\ m & o \end{pmatrix} B_n^\dagger |0\rangle, \qquad (5.1)$$

since $D_{mi} |0\rangle = 0$. This leads to

$$\langle \phi_0 | \phi_0 \rangle = 2 - 2\lambda \begin{pmatrix} o & o \\ o & o \end{pmatrix}, \qquad (5.2)$$

with $\lambda \begin{pmatrix} o & o \\ o & o \end{pmatrix} = (33\pi/2)(a_B/L)^3$ for three-dimensional (3D) cobosons[28] with zero center-of-mass momentum and relative motion wave function $\langle r | \nu_0 \rangle = e^{-r/a_B}/\sqrt{\pi a_B^3}$, so that this Pauli scattering goes to zero when the sample size L increases.

Using Eq. (3.16) and the fact that $V_0^\dagger |0\rangle = 0$, we find

$$H |\phi_0\rangle = 2E_o B_o^{\dagger 2} |0\rangle - \sum \xi \begin{pmatrix} n & o \\ m & o \end{pmatrix} B_m^\dagger B_n^\dagger |0\rangle. \qquad (5.3)$$

From Eqs. (3.6) and (3.7), it is easy to show that the scalar product of two coboson states reads as[31]

$$\langle 0 | B_m B_n B_i^\dagger B_j^\dagger |0\rangle = \left\{ \delta_{mi} \delta_{nj} - \lambda \begin{pmatrix} n & i \\ m & j \end{pmatrix} \right\} + \{m \leftrightarrow n\}.$$

(5.4)

So that, from the two above equations, we end up with

$$\frac{\langle \phi_0 | H | \phi_0 \rangle}{\langle \phi_0 | \phi_0 \rangle} = 2E_o + \frac{\xi \begin{pmatrix} o & o \\ o & o \end{pmatrix} - \xi^{\rm in} \begin{pmatrix} o & o \\ o & o \end{pmatrix}}{1 - \lambda \begin{pmatrix} o & o \\ o & o \end{pmatrix}}$$

$$= 2E_o + \frac{\hat{\xi}_{oo;oo}}{1 - \lambda_{oo;oo}} \approx 2E_o + \hat{\xi}_{oo;oo} \qquad (5.5)$$

since $\lambda_{oo;oo}$ goes to 0 as $(a_B/L)^d$ when the sample size increases.

B. Fragmented state

We now turn to the fragmented state $|\phi_{12}\rangle = B^{\dagger}_{o_1} B^{\dagger}_{o_2} |0\rangle$ for (o_1, o_2) with different center-of-mass momenta in order for the Pauli and interaction scatterings to cancel due to momentum conservation if the number of o_1 cobosons in the in and out states are different. Using Eq. (5.4), its norm is found to be

$$\langle \phi_{12} | \phi_{12} \rangle = 1 - \lambda \begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix} - \lambda \begin{pmatrix} o_1 & o_2 \\ o_2 & o_1 \end{pmatrix} \quad (5.6)$$

$$= 1 - 2\lambda_{o_1 o_2 ; o_1 o_2}, \quad (5.7)$$

while from Eqs. (3.8) and (3.9), the Hamiltonian mean value in this fragmented state appears as

$$\frac{\langle \phi_{12} | H | \phi_{12} \rangle}{\langle \phi_{12} | \phi_{12} \rangle} = E_{o_1} + E_{o_2} + \frac{\xi \begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix} - \xi^{\mathrm{in}} \begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix} + \xi \begin{pmatrix} o_1 & o_2 \\ o_2 & o_1 \end{pmatrix} - \xi^{\mathrm{in}} \begin{pmatrix} o_1 & o_2 \\ o_2 & o_1 \end{pmatrix}}{1 - 2\lambda_{o_1 o_2 ; o_1 o_2}}. \quad (5.8)$$

So that if we use the fully symmetrized scattering defined in Eq. (3.13) and take a large sample volume in order for the λ term to give a negligible contribution, this Hamiltonian mean value reduces, for $(o_1, o_2) \approx o$, to

$$\frac{\langle \phi_{12} | H | \phi_{12} \rangle}{\langle \phi_{12} | \phi_{12} \rangle} = E_{o_1} + E_{o_2} + 2 \frac{\hat{\xi}_{o_1 o_2 ; o_1 o_2}}{1 - \lambda_{1 o_2 ; o_1 o_2}} \approx 2 E_o + 2 \hat{\xi}_{oo ; oo}. \quad (5.9)$$

C. Coherent superposition

The third state of interest is the coherent superposition $|\phi\rangle = B^{\dagger 2} |0\rangle$, where $B^{\dagger} = a' B^{\dagger}_{o'} + a'' B^{\dagger}_{o''}$, with $|a'|^2 + |a''|^2 = 1$ and $\vec{Q}_{o'} \neq \vec{Q}_{o''}$ in order for the Pauli and interaction scatterings to again cancel due to momentum conservation if the number of o' cobosons in the in and out states are different. As a direct consequence, $\langle \phi | \phi \rangle$ and $\langle \phi | H | \phi \rangle$ can only have terms in $|a'|^4$, $|a''|^4$, and $|a'a''|^2$. Using Eq. (5.4), the bare expansion of $B^{\dagger 2}$ in terms of $B^{\dagger 2}_{o'}$, $B^{\dagger 2}_{o''}$, and $B^{\dagger}_{o'} B^{\dagger}_{o''}$ readily leads to the norm of the coherent state given by

$$\langle \phi | \phi \rangle = |a'|^4 L_{o'} + |a''|^4 L_{o''} + |a'a''|^2 L_{o',o''}, \quad (5.10)$$

with $L_{o'} = 2 - 2\lambda \begin{pmatrix} o' & o' \\ o' & o' \end{pmatrix}$ and similarly for $L_{o''}$, while $L_{o',o''} = 1 - \lambda \begin{pmatrix} o'' & o'' \\ o' & o' \end{pmatrix} - \lambda \begin{pmatrix} o' & o'' \\ o'' & o' \end{pmatrix}$. In terms of the symmetrized Pauli scatterings defined as in Eq. (3.13), this norm reduces to

$$\langle \phi | \phi \rangle = 2 - 2|a'|^4 \lambda_{o'o';o'o'} - 2|a''|^4 \lambda_{o''o'';o''o''}$$
$$- 8|a'a''|^2 \lambda_{o'o'';o'o''}. \quad (5.11)$$

In the same way, Eqs. (3.8) and (3.9) allow us to write

$$\langle \phi | H | \phi \rangle = |a'|^4 A_{o'} + |a''|^4 A_{o''} + 4|a'a''|^2 A_{o',o''}, \quad (5.12)$$

where $A_{o'} = 2E_{o'}[2 - 2\lambda \begin{pmatrix} o' & o' \\ o' & o' \end{pmatrix}] + 2\hat{\xi} \begin{pmatrix} o' & o' \\ o' & o' \end{pmatrix}$ and similarly for $A_{o''}$, while

$$A_{o',o''} = (E_{o'} + E_{o''}) \left[1 - \lambda \begin{pmatrix} o'' & o'' \\ o' & o' \end{pmatrix} - \lambda \begin{pmatrix} o' & o'' \\ o'' & o' \end{pmatrix} \right]$$
$$+ \hat{\xi} \begin{pmatrix} o'' & o'' \\ o' & o' \end{pmatrix} + \hat{\xi} \begin{pmatrix} o' & o'' \\ o'' & o' \end{pmatrix}.$$

For $o' \approx o'' \approx o$ in a large sample volume, the expectation value of the Hamiltonian then reduces to

$$\frac{\langle \phi | H | \phi \rangle}{\langle \phi | \phi \rangle} \approx 2 E_o + \frac{(1 + 2|a'a''|^2) \hat{\xi}_{oo;oo}}{1 - (1 + 2|a'a''|^2) \lambda_{oo;oo}}$$
$$\approx 2 E_o + (1 + 2|a'a''|^2) \hat{\xi}_{oo;oo}. \quad (5.13)$$

D. Effective scattering for bosonized particles

When compared to similar results for elementary bosons, namely Eqs. (4.3), (4.5), and (4.7), the Hamiltonian mean values in the three two-coboson states, obtained above, lead us to identify the diagonal effective scattering for elementary bosons $\bar{\xi}_{oo;oo}$ with the physically relevant combination of energylike scatterings defined in Eq. (3.12), namely,

$$\bar{\xi}_{oo;oo} \equiv \hat{\xi}_{oo;oo} = \xi \begin{pmatrix} o & o \\ o & o \end{pmatrix} - \xi^{\mathrm{in}} \begin{pmatrix} o & o \\ o & o \end{pmatrix}. \quad (5.14)$$

This effective scattering contains a direct contribution as well as an exchange contribution which is symmetrical with respect to the in and out processes, as physically reasonable for $\xi^{\mathrm{in}} \begin{pmatrix} o & o \\ o & o \end{pmatrix} = \xi^{\mathrm{out}} \begin{pmatrix} o & o \\ o & o \end{pmatrix}$ due to Eq. (3.11).

For excitons or H atoms,[28] the diagonal direct scattering $\xi \begin{pmatrix} o & o \\ o & o \end{pmatrix}$ reduces to 0, the repulsion between fermions α or fermions β being as large as the attraction between (α, β). On the opposite, the diagonal exchange scattering of these cobosons differ from zero: in three dimensions, it reads $\xi^{\mathrm{in}} \begin{pmatrix} o & o \\ o & o \end{pmatrix} = -(26\pi/3) R_0 (a_B/L)^3$, where $R_0 = \mu e^4 / 2\hbar^2 \varepsilon^2 = e^2 / 2a_B$ is the coboson Rydberg energy.[28]

VI. MANY ELEMENTARY BOSONS

We now turn to states with a large number of bosons and first consider that these bosons are elementary bosons.

A. Single state

Let us start with the state with its N bosons in a single state, $|\bar{\phi}_0\rangle = \bar{B}_o^{\dagger N}|0\rangle$. We physically expect the interaction term of the Hamiltonian expectation value to depend on the boson number through $N(N-1)/2$ which corresponds to the number of interacting boson pairs (o,o) we can form out of N bosons in state o. This leads us to expect that the Hamiltonian expectation value obtained for $N=2$, as given in Eq. (4.3), must transform for general N as

$$\frac{\langle \bar{\phi}_0 \bar{H} | \bar{\phi}_0 \rangle}{\langle \bar{\phi}_0 | \bar{\phi}_0 \rangle} = N E_o + \frac{N(N-1)}{2} \bar{\xi}_{oo;oo}. \qquad (6.1)$$

For scatterings $\xi_{oo;oo}$ in $(a_B/L)^d$, this will induce a correction to the bare energy $N E_o$ on the order of $N\eta$, where η is the dimensionless parameter associated to the density given in Eq. (1.4). Let us now recover this physically expected result.

Equation (2.4) readily gives the well-known normalization factor for elementary bosons, namely,

$$\langle \bar{\phi}_0 | \bar{\phi}_0 \rangle = \langle 0 | \bar{B}_o^{N-1} \bar{B}_o \bar{B}_o^{\dagger N} | 0 \rangle = N \langle 0 | \bar{B}_o^{N-1} \bar{B}_o^{\dagger N-1} | 0 \rangle = N!. \qquad (6.2)$$

Using the same equation, Eq. (2.4), on the one-body part of the Hamiltonian leads to

$$\begin{aligned}
\langle \bar{\phi}_0 | \bar{H}_0 | \bar{\phi}_0 \rangle &= \sum E_i \{ \langle \bar{\phi}_0 | \bar{B}_i^{\dagger} \} \{ \bar{B}_i | \bar{\phi}_0 \rangle \} \\
&= N^2 E_o (N-1)! \\
&= N E_o \langle \bar{\phi}_0 | \bar{\phi}_0 \rangle,
\end{aligned} \qquad (6.3)$$

while on the two-body part, we find

$$\begin{aligned}
\langle \bar{\phi}_0 | \bar{V} | \bar{\phi}_0 \rangle &= \frac{1}{2} \sum \bar{\xi} \begin{pmatrix} n & j \\ m & i \end{pmatrix} \{ \langle \bar{\phi}_0 | \bar{B}_m^{\dagger} \bar{B}_n^{\dagger} \} \{ \bar{B}_j \bar{B}_i | \bar{\phi}_0 \rangle \} \\
&= \frac{1}{2} N^2 (N-1)^2 \bar{\xi} \begin{pmatrix} o & o \\ o & o \end{pmatrix} (N-2)!.
\end{aligned} \qquad (6.4)$$

This just proves that the guess of Eq. (6.1) is correct.

B. Fragmented state

We now consider the fragmented state $|\bar{\phi}_{12}\rangle = \bar{B}_{o_1}^{\dagger N_1} \bar{B}_{o_2}^{\dagger N_2} |0\rangle$. The same physical understanding leads us to expect the interaction term between elementary bosons (o_1, o_2) of the Hamiltonian expectation value, as given in Eq. (4.5), to appear with a $N_1 N_2$ prefactor. Since we now have many bosons o_1 and many bosons o_2, we should also have an interaction term between bosons o_1 with a prefactor $N_1(N_1-1)/2$ and an interaction term between bosons o_2 with a prefactor $N_2(N_2-1)/2$. From Eq. (4.5), we are thus led to guess

$$\begin{aligned}
\frac{\langle \bar{\phi}_{12} | \bar{H} | \bar{\phi}_{12} \rangle}{\langle \bar{\phi}_{12} | \bar{\phi}_{12} \rangle} &= N_1 E_{o_1} + N_2 E_{o_2} + \frac{N_1(N_1-1)}{2} \bar{\xi}_{o_1 o_1; o_1 o_1} \\
&\quad + \frac{N_2(N_2-1)}{2} \bar{\xi}_{o_2 o_2; o_2 o_2} + 2 N_1 N_2 \bar{\xi}_{o_1 o_2; o_1 o_2}.
\end{aligned} \qquad (6.5)$$

For $o_1 \approx o_2 \approx o$, the Hamiltonian expectation value should thus be given by

$$\frac{\langle \bar{\phi}_{12} | \bar{H} | \bar{\phi}_{12} \rangle}{\langle \bar{\phi}_{12} | \bar{\phi}_{12} \rangle} \simeq N E_o + \left(\frac{N(N-1)}{2} + N_1 N_2 \right) \bar{\xi}_{oo;oo} \quad (6.6)$$

for $N_1 + N_2 = N$. Let us now show this result explicitly.

Equation (2.4) leads to

$$\bar{B}_i | \bar{\phi}_{12} \rangle = (N_1 \delta_{i o_2} \bar{B}_{o_1}^{\dagger N_1 - 1} \bar{B}_{o_2}^{\dagger N_2} + N_2 \delta_{i o_2} \bar{B}_{o_1}^{\dagger N_1} \bar{B}_{o_2}^{\dagger N_2 - 1}) | 0 \rangle, \qquad (6.7)$$

so that for $o_1 \neq o_2$, we find $\bar{B}^{N_1} | \bar{\phi}_{12} \rangle = N_1! \bar{B}_{o_2}^{\dagger N_2} | 0 \rangle$. This readily gives $\langle \bar{\phi}_{12} | \bar{\phi}_{12} \rangle = N_1! N_2!$, while

$$\begin{aligned}
\langle \bar{\phi}_{12} | \bar{H}_0 | \bar{\phi}_{12} \rangle &= \sum E_i \langle \bar{\phi}_{12} | \bar{B}_i^{\dagger} \bar{B}_i | \bar{\phi}_{12} \rangle \\
&= E_{o_1} N_1^2 (N_1-1)! N_2! + E_{o_2} N_2^2 (N_2-1)! N_1!.
\end{aligned} \qquad (6.8)$$

If we now use Eq. (6.7) to calculate $\bar{B}_j \bar{B}_i | \bar{\phi}_{12} \rangle$, the scalar product $\{ \langle \bar{\phi}_{12} | \bar{B}_m^{\dagger} \bar{B}_n^{\dagger} \} \{ \bar{B}_j \bar{B}_i | \bar{\phi}_{12} \rangle \}$ which appears in $\langle \bar{\phi}_{12} | \bar{V} | \bar{\phi}_{12} \rangle$ leads to

$$\langle \bar{\phi}_{12} | \bar{V} | \bar{\phi}_{12} \rangle = \frac{1}{2} (\bar{W}_{o_1} + \bar{W}_{o_2}) + \bar{W}_{o_1, o_2}, \qquad (6.9)$$

in which we have set

$$\bar{W}_{o_1} = \bar{\xi} \begin{pmatrix} o_1 & o_1 \\ o_1 & o_1 \end{pmatrix} [N_1(N_1-1)]^2 (N_1-2)! N_2!, \quad (6.10)$$

and similarly for $\bar{W}_{o_2}$, while

$$\begin{aligned}
\bar{W}_{o_1, o_2} &= \left[\bar{\xi} \begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix} + \bar{\xi} \begin{pmatrix} o_1 & o_2 \\ o_2 & o_1 \end{pmatrix} \right] \\
&\quad \times (N_1 N_2)^2 (N_1-1)! (N_2-1)!.
\end{aligned} \qquad (6.11)$$

The expected result, Eq. (6.5), then readily follows from Eqs. (6.8) and (6.9).

C. Coherent superposition

The third state of interest is the coherent superposition of states $|\bar{\phi}\rangle = \bar{B}^{\dagger N} |0\rangle$ with $\bar{B}^{\dagger} = a' \bar{B}_{o'}^{\dagger} + a'' \bar{B}_{o''}^{\dagger}$. Through similar physical arguments, we expect to have the interaction term in the energy for $N=2$ cobosons, as given in Eq. (4.7), to appear with a prefactor $N(N-1)/2$. To show it explicitly, we use Eq. (2.6) to get

$$\langle\bar{\phi}|\bar{\phi}\rangle = \langle0|\bar{B}^{N-1}(a'^{*}\bar{B}_{o'} + a''^{*}\bar{B}_{o''})\bar{B}^{\dagger N}|0\rangle$$

$$= N[|a'|^{2} + |a''|^{2}]\langle0|\bar{B}^{N-1}\bar{B}^{\dagger N-1}|0\rangle. \quad (6.12)$$

Its iteration, for $|a'|^{2}+|a''|^{2}=1$, gives $\langle\bar{\phi}|\bar{\phi}\rangle=N!$ as for a condensate made of a single state. In the same way, the norm of $\bar{B}_{i}|\bar{\phi}\rangle$ deduced from Eq. (2.6), leads to

$$\langle\bar{\phi}|\bar{H}_{0}|\bar{\phi}\rangle = N(|a'|^{2}E_{o'} + |a''|^{2}E_{o''})N!, \quad (6.13)$$

while the scalar product of the states $\langle\bar{\phi}|\bar{B}_{m}^{\dagger}\bar{B}_{n}^{\dagger}$ and $\bar{B}_{i}\bar{B}_{j}|\bar{\phi}\rangle$ leads, for $\vec{Q}_{o'} \neq \vec{Q}_{o''}$, to

$$\langle\bar{\phi}|\bar{V}|\bar{\phi}\rangle = \frac{1}{2}[N(N-1)]^{2}(N-2)!\left\{|a'|^{4}\bar{\xi}\begin{pmatrix}o' & o'\\ o' & o'\end{pmatrix}\right.$$
$$+ |a''|^{2}\bar{\xi}\begin{pmatrix}o'' & o''\\ o'' & o''\end{pmatrix} + 2|a'a''|^{2}\left[\bar{\xi}\begin{pmatrix}o'' & o''\\ o' & o'\end{pmatrix}\right.$$
$$\left.\left.+ \bar{\xi}\begin{pmatrix}o' & o''\\ o'' & o'\end{pmatrix}\right]\right\}, \quad (6.14)$$

which also reads

$$\langle\bar{\phi}|\bar{V}|\bar{\phi}\rangle = \frac{N(N-1)}{2}[|a'|^{4}\bar{\xi}_{o'o';o'o'} + |a''|^{2}\bar{\xi}_{o''o'';o''o''}$$
$$+ 4|a'a''|^{2}\bar{\xi}_{o'o'';o'o''}]N!.$$

So that for $o' \approx o'' \approx o'$, we end with the expected result, namely,

$$\frac{\langle\bar{\phi}|H|\bar{\phi}\rangle}{\langle\bar{\phi}|\bar{\phi}\rangle} \simeq NE_{o} + \frac{N(N-1)}{2}(1 + 2|a'a''|^{2})\bar{\xi}_{oo;oo}.$$

$$(6.15)$$

VII. MANY COMPOSITE BOSONS

A. Single state

The Hamiltonian mean value for the pure state $|\phi_{0}\rangle = B_{o}^{\dagger N}|0\rangle$ was already calculated in a previous work.[28,36] It has a naive contribution NE_{o}. It also has a set of density-dependent corrections in η^{n} with $n\geq1$, in contrast with elementary bosons which only have a $n=1$ term. This set of density terms comes from fermion exchanges between the N cobosons. Since the Hamiltonian expectation value $\langle H\rangle$ only contains one interaction by construction, the density terms for $n\geq2$ can only come from the fermion exchanges between three or more cobosons induced by the Pauli exclusion principle. They are nicely visualized by Shiva diagrams with $n+1$ cobosons and *one* interaction process between any two of these coboson lines (see Fig. 6). Let us here repeat the main steps of this calculation for completeness—and also because the ones for $|\phi_{12}\rangle$ and $|\phi\rangle$ are conceptually similar while far more complex.

In order to calculate $\langle\phi_{0}|H|\phi_{0}\rangle$, we use one of the two key equations for coboson many-body effects, namely, Eq. (3.16), to find
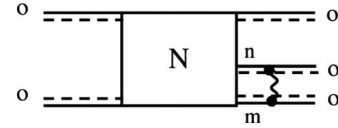

FIG. 6. Shiva diagram for a process having one fermion interaction between N cobosons o, as given by the second term in Eq. (7.1).

$$\langle\phi_{0}|H|\phi_{0}\rangle = NE_{o}\langle\phi_{0}|\phi_{0}\rangle + \frac{N(N-1)}{2}\sum\xi\begin{pmatrix}n & o\\ m & o\end{pmatrix}$$
$$\times\langle0|B_{o}^{N}B_{o}^{\dagger N-2}B_{m}^{\dagger}B_{n}^{\dagger}|0\rangle. \quad (7.1)$$

The second term of this equation is shown in Fig. 6. In it, appears the scalar product of N coboson states with two cobosons different from o on the right (see Fig. 7). This scalar product is calculated using another key equation for many-body effects, namely, Eq. (3.14). This equation leads to
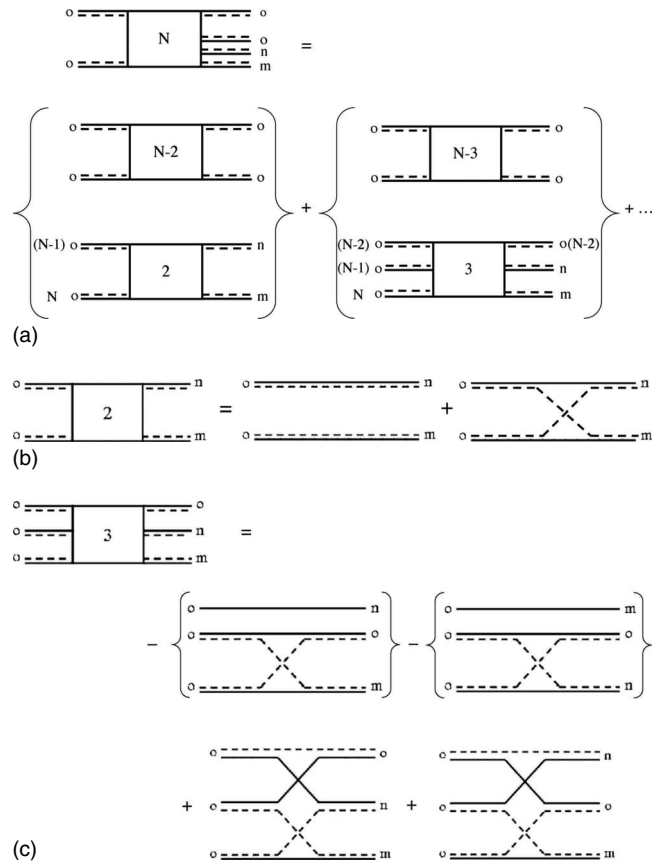

FIG. 7. (a) Shiva diagram for the scalar product of N coboson states appearing in Eq. (7.2), with N cobosons o on the left and $(N-2)$ cobosons o plus two cobosons (m,n) on the right. The standard way (Refs. 28 and 31) to calculate this scalar product is to isolate $(N-2),(N-3),\ldots$ cobosons o not involved in fermion exchanges with (m,n). The N prefactors come from the number of ways to choose the cobosons o having fermion exchanges with (m,n). These possible exchanges are shown in (b) and (c).

$$\langle 0|B_o^{N-2}B_m B_n B_o^{\dagger N}|0\rangle = \langle 0|B_o^{N-2}B_m \left[N\delta_{no}B_0^{\dagger N-1} \right.$$
$$\left. - N(N-1)B_o^{\dagger N-2}\sum \lambda \begin{pmatrix} p & o \\ n & o \end{pmatrix} B_p^\dagger \right]|0\rangle.$$
$$(7.2)$$

We then use the same commutation, Eq. (3.14), to pass B_m over $B_o^{\dagger N-1}$ and $B_o^{\dagger N-2}$. Since $[\lambda\begin{pmatrix} n & i \\ m & j \end{pmatrix}]^* = \lambda\begin{pmatrix} j & n \\ i & m \end{pmatrix}$, this allows us to expand the above scalar product as (see Fig. 7)

$$\langle 0|B_o^N B_m^\dagger B_n^\dagger B_o^{\dagger N-2}|0\rangle$$
$$= N(N-1)\left[\delta_{on}\delta_{om} - \lambda\begin{pmatrix} o & n \\ o & m \end{pmatrix} \right](N-2)!\,F_{N-2}$$
$$+ N(N-1)(N-2)^2\left[-\delta_{on}\lambda\begin{pmatrix} o & o \\ o & m \end{pmatrix} - \delta_{om}\lambda\begin{pmatrix} o & o \\ o & m \end{pmatrix} \right.$$
$$\left. + \lambda_3 + \lambda_3' \right](N-3)!\,F_{N-2} + \cdots, \quad (7.3)$$

where λ_3 and λ_3' are three-leg scatterings shown in Fig. 7(c),

$$\lambda_3 = \lambda\begin{pmatrix} o & o \\ o & n \\ o & m \end{pmatrix}, \quad \lambda_3' = \lambda\begin{pmatrix} o & n \\ o & o \\ o & m \end{pmatrix},$$

in which the cobosons (m,n) exchange their fermions with one coboson o to produce three cobosons o. This expansion actually follows the standard procedure[31] to calculate scalar products, namely, we first isolate terms in

$$\langle 0|B_o^{N-P}B_o^{\dagger N-P}|0\rangle = (N-P)!\,F_{N-P},$$

with $P \geq 2$. We then connect the remaining cobosons in all possible ways while enforcing the cobosons o to be "never alone" as in Figs. 7(b) and 7(c). The N prefactors in Fig. 7(a) correspond to the number of ways we can choose the cobosons o among N on the left and among $(N-2)$ on the right. In the case of the first term in this figure, we just have to choose the two cobosons o on the left; this is why this first term appears with a prefactor $N(N-1)$. In the second term, we also have to choose a coboson o on the right and a third coboson o on the left. This is why this term appears with a prefactor $[N(N-1)(N-2)](N-2)$, and so on.

By inserting Eq. (7.3) into Eq. (7.1), we readily find

$$\frac{\langle \phi_0|H|\phi_0\rangle}{\langle \phi_0|\phi_0\rangle} = NE_o + \frac{F_{N-2}}{F_N}\frac{N(N-1)}{2}\left[\xi\begin{pmatrix} o & o \\ o & o \end{pmatrix} - \xi^{in}\begin{pmatrix} o & o \\ o & o \end{pmatrix} \right]$$
$$+ \cdots, \quad (7.4)$$

so that, since $F_{N-2}/F_N = (F_{N-2}/F_{N-1})(F_{N-1}/F_N) = 1 + \mathcal{O}(\eta)$, we end for large N with

$$\frac{\langle \phi_0|H|\phi_0\rangle}{\langle \phi_0|\phi_0\rangle} = N\left[E_o + \frac{N}{2}\hat{\xi}_{oo;oo} + \mathcal{O}(\eta^2) \right]. \quad (7.5)$$

This shows that the first correction to the bare energy E_o is of order $\eta = N(a_B/L)^d$, because, as shown in Sec. V D, $\hat{\xi}_{oo;oo}$ is on the order of $(a_B/L)^d$. Note that as for the Coulomb interaction $\xi\begin{pmatrix} o & o \\ o & o \end{pmatrix} = 0$, the effective scattering $\hat{\xi}_{oo;oo}$ reduces to

$-\xi^{in}\begin{pmatrix} o & o \\ o & o \end{pmatrix}$ which for 3D excitons is equal to $-(26\pi/3)R_0(a_B/L)^3$. Consequently, this Hamiltonian mean value is just the energy of N electron-hole pairs obtained by Keldysh and Kozlov,[16] using a completely different approach in which these pairs are not treated as coboson entities, as we do here.

B. Coherent superposition

In order to calculate the mean value of the Hamiltonian for the N coherent cobosons $|\phi\rangle = B^{\dagger N}|0\rangle$, we use the results derived in Sec. III C. From Eq. (3.30), this mean value appears as

$$\langle 0|B^N H B^{\dagger N}|0\rangle = NE_o\langle 0|B^N \widetilde{B}^\dagger B^{\dagger N-1}|0\rangle + \frac{N(N-1)}{2}\Delta, \quad (7.6)$$

$$\Delta = \sum \xi_{mn}\langle 0|B^N B_m^\dagger B_n^\dagger B^{\dagger N-2}|0\rangle, \quad (7.7)$$

where ξ_{mn} is given in Eq. (3.28). The first term of Eq. (7.6) reduces to $NE_o\langle\phi|\phi\rangle$ for $E_{o'} \approx E_{o''} \approx E_o$ since $\widetilde{B}^\dagger \approx B^\dagger$. To calculate Δ, we use Eq. (3.26) for $\langle 0|B^N B_m^\dagger$. This leads to

$$\Delta = N\sum \xi_{mn}\delta_m^*\langle 0|B^{N-1}B_n^\dagger B^{\dagger N-2}|0\rangle$$
$$- N(N-1)\sum \xi_{mn}\lambda_{mp}^*\langle 0|B^{N-2}B_p B_n^\dagger B^{\dagger N-2}|0\rangle. \quad (7.8)$$

In the first term, we again use Eq. (3.23) for $\langle 0|B^{N-1}B_n^\dagger$, while we use the commutator $[B_p, B_n^\dagger]$ given in Eq. (3.6) to calculate the second term of Δ. This leads to

$$\Delta = N(N-1)(N-2)!\,G_{N-2}\sum \xi_{mn}(\delta_m^*\delta_n^* - \lambda_{mn}^*) + \cdots G_{N-3}$$
$$+ \cdots. \quad (7.9)$$

If we now use the definitions of δ_m, ξ_{mn}, and λ_{mn} given in Eqs. (2.7), (3.21), and (3.28), we find that for $\vec{Q}_{o'} \neq \vec{Q}_{o''}$, the sum in Eq. (7.9) reduces to

$$|a'|^4\hat{\xi}\begin{pmatrix} o' & o' \\ o' & o' \end{pmatrix} + |a''|^4\hat{\xi}\begin{pmatrix} o'' & o'' \\ o'' & o'' \end{pmatrix}$$
$$+ 2|a'a''|^2\left\{ \hat{\xi}\begin{pmatrix} o'' & o'' \\ o' & o' \end{pmatrix} + \hat{\xi}\begin{pmatrix} o'' & o' \\ o' & o'' \end{pmatrix} \right\},$$

with $\hat{\xi} = \xi - \xi^{in}$ as defined in Eq. (3.12). By collecting all the terms, we end with

$$\frac{\langle \phi|H|\phi\rangle}{\langle \phi|\phi\rangle} \simeq NE_o + \frac{N(N-1)}{2}\frac{G_{N-2}}{G_N}\{|a'|^4\hat{\xi}_{o'o';o'o'}$$
$$+ |a''|^4\hat{\xi}_{o''o'';o''o''} + 4|a'a''|^2\hat{\xi}_{o'o'';o'o''}\} + \cdots \quad (7.10)$$

within corrections on the order of $(E_{o'} - E_{o''})$. For $o' \approx o'' \approx o$ and N large, the above result reduces to

$$\frac{\langle \phi|H|\phi\rangle}{\langle \phi|\phi\rangle} = N\left[E_o + \frac{N}{2}(1 + 2|a'a''|^2)\hat{\xi}_{oo;oo} + O|\eta^2 \right]. \quad (7.11)$$
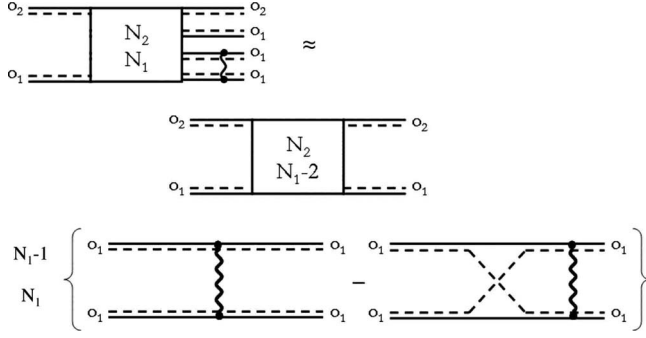
FIG. 8. Shiva diagrams for Δ_{11} defined in Eq. (7.14), in which two cobosons o_1 among N_1 have an interaction scattering, the other N_2 cobosons o_2 just possibly exchanging their fermions with the cobosons o_1. The N prefactors are the numbers of ways to choose the cobosons o_1 on the left.

C. Fragmented state

The last N-coboson state we must consider, $|\phi_{12}\rangle = B_{o_1}^{\dagger N_1} B_{o_2}^{\dagger N_2} |0\rangle$, has two large numbers of different cobosons in states o_1 and o_2. The very many exchanges which exist within the o_1 population, within the o_2 population, and between the o_1 and o_2 populations make this many-body calculation quite tricky. This is why we have kept it for the end, the previous ones having the role of useful exercises. In view of the above results, we can however guess that the Hamiltonian mean value for composite bosons in this fragmented state should read as the one for elementary bosons, namely, Eq. (6.5), with, according to Eq. (5.14), $\overline{\xi}_{o_1 o_1 ; o_1 o_1}$ and $\overline{\xi}_{o_1 o_2 ; o_1 o_2}$ replaced by $\hat{\xi}_{o_1 o_1 ; o_1 o_1}$ and $\hat{\xi}_{o_1 o_2 ; o_1 o_2}$, respectively. Let us show this explicitly.

We first use Eq. (3.16) in Sec. III B to replace $HB_{o_1}^{\dagger N_1}$ in $H|\phi_{12}\rangle$. This leads to

$$
H|\phi_{12}\rangle = \left\{ B_{o_1}^{N_1} H + N_1 E_{o_1} B_{o_1}^{\dagger N_1} + N_1 B_{o_1}^{\dagger N_1 - 1} V_{o_1}^{\dagger} \right.
$$
$$
\left. + \frac{N_1(N_1 - 1)}{2} B_{o_1}^{\dagger N_1 - 2} \sum \xi \begin{pmatrix} n & o_1 \\ m_1 & o_1 \end{pmatrix} B_m^{\dagger} B_n^{\dagger} \right\} B_{o_2}^{\dagger N_2} |0\rangle.
$$
(7.12)

We again use Eq. (3.16) for $HB_{o_2}^{\dagger N_2}|0\rangle$ and Eq. (3.17) for $V_{o_1}^{\dagger} B_{o_2}^{\dagger N_2}|0\rangle$. This allows to split $\langle \phi_{12}|H|\phi_{12}\rangle$ into

$$
\langle \phi_{12}|H|\phi_{12}\rangle = (N_1 E_{o_1} + N_2 E_{o_2})\langle \phi_{12}|\phi_{12}\rangle + \frac{N_1(N_1-1)}{2}\Delta_{11}
$$
$$
+ \frac{N_2(N_2-1)}{2}\Delta_{22} + N_1 N_2 \Delta_{12}.
$$
(7.13)

Δ_{11}, shown in Fig. 8, describes the interactions of two among N_1 cobosons in state o_1, the other N_2 cobosons o_2 possibly having fermion exchanges with the cobosons o_1. The precise value of Δ_{11} is
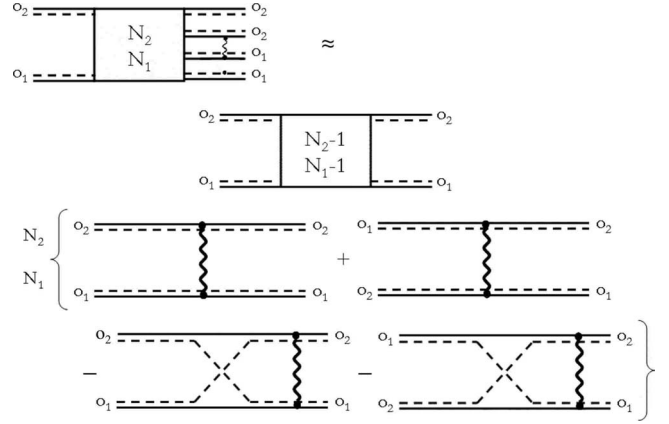

FIG. 9. Same as Fig. 8, for Δ_{12} defined in Eq. (7.15), the interaction taking place between one coboson o_1 among N_1 and one coboson o_2 among N_2. The N prefactors are the number of ways to choose the cobosons (o_1, o_2) on the left.

$$
\Delta_{11} = \sum \langle 0| B_{o_2}^{N_2} B_{o_1}^{N_1} B_m^{\dagger} B_n^{\dagger} B_{o_1}^{\dagger N_1 - 2} B_{o_2}^{\dagger N_2} |0\rangle \xi \begin{pmatrix} n & o_1 \\ m & o_1 \end{pmatrix},
$$
(7.14)

and similarly for Δ_{22}, while Δ_{12}, shown in Fig. 9, results from the interaction of one coboson o_1 with one coboson o_2. It reads

$$
\Delta_{12} = \sum \langle 0| B_{o_2}^{N_2} B_{o_1}^{N_1} B_m^{\dagger} B_n^{\dagger} B_{o_1}^{\dagger N_1 - 1} B_{o_2}^{\dagger N_2 - 1} |0\rangle \xi \begin{pmatrix} n & o_2 \\ m & o_1 \end{pmatrix}.
$$
(7.15)

The reader who is knowledgeable about Shiva diagrams[28,31] will immediately see from Fig. 8 that for $\vec{Q}_{o_1} \neq \vec{Q}_{o_2}$, i.e., for $\xi \begin{pmatrix} o_2 & o_1 \\ o_1 & o_1 \end{pmatrix} = 0 = \xi^{in} \begin{pmatrix} o_2 & o_1 \\ o_1 & o_1 \end{pmatrix}$, we must have

$$
\Delta_{11} \simeq N_1(N_1-1) \left[\xi \begin{pmatrix} o_1 & o_1 \\ o_1 & o_1 \end{pmatrix} - \xi^{in} \begin{pmatrix} o_1 & o_1 \\ o_1 & o_1 \end{pmatrix} \right]
$$
$$
\times (N_1 - 2)! \, N_2! \, G_{N_1 - 2, N_2},
$$
(7.16)

and similarly for Δ_{22}. Here G_{N_1, N_2} is defined as F_N, through the norm of the $B_{o_1}^{\dagger N_1} B_{o_2}^{\dagger N_2}|0\rangle$ state [see Eq. (3.36)].

In the same way, the Shiva diagram in Fig. 9 readily leads to, for $\vec{Q}_{o_1} \neq \vec{Q}_{o_2}$,

$$
\Delta_{12} \simeq N_1 N_2 \left[\xi \begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix} + \xi \begin{pmatrix} o_1 & o_2 \\ o_2 & o_1 \end{pmatrix} - \xi^{in} \begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix} \right.
$$
$$
\left. - \xi^{in} \begin{pmatrix} o_1 & o_2 \\ o_2 & o_1 \end{pmatrix} \right] (N_1 - 1)! \, (N_2 - 1)! \, G_{N_1 - 1, N_2 - 1}.
$$
(7.17)

By collecting all these terms, we end with

$$\frac{\langle\phi_{12}|H|\phi_{12}\rangle}{\langle\phi_{12}|\phi_{12}\rangle} \simeq N_1 E_{o_1} + N_2 E_{o_2} + \frac{N_1(N_1-1)}{2}\hat{\xi}_{o_1 o_1; o_1 o_1}\frac{G_{N_1-2,N_2}}{G_{N_1,N_2}}$$

$$+ \frac{N_2(N_2-1)}{2}\hat{\xi}_{o_2 o_2; o_2 o_2}\frac{G_{N_1,N_2-2}}{G_{N_1,N_2}}$$

$$+ 2N_1 N_2 \hat{\xi}_{o_1 o_2; o_1 o_2}\frac{G_{N_1-1,N_2-1}}{G_{N_1,N_2}}. \tag{7.18}$$

Like F_N, the factor G_{N_1,N_2}, which comes from the many fermion exchanges which take place between the N_1 cobosons in state o_1 and the N_2 cobosons in state o_2, is exponentially small. However the ratios of G_{N_1,N_2}'s are nearly 1 at lowest order in density. Enforcing $N_1+N_2=N$ and $o_1 \approx o_2 \approx o$, we ultimately find the expected result, namely,

$$\frac{\langle\phi_{12}|H|\phi_{12}\rangle}{\langle\phi_{12}|\phi_{12}\rangle} \simeq N E_o + \left(\frac{N(N-1)}{2} + N_1 N_2\right)\hat{\xi}_{oo;oo}. \tag{7.19}$$

Let us end this section by returning to the interaction parts Δ_{11}, Δ_{22}, and Δ_{12} of the Hamiltonian mean value defined in Eqs. (7.14) and (7.15) and by calculating them, not through Shiva diagrams, but in a pedestrian way using the commutators appropriate to many-body effects, namely, Eqs. (3.14) and (3.15). The problem is to get the scalar products of coboson states which appear in these Δ's. This always is the tricky part of any calculation involving cobosons. The ones of interest here are

$$S^{(P)} = \langle 0|B_{o_2}^{N_2}B_{o_1}^{N_1}B_m^\dagger B_n^\dagger B_{o_1}^{\dagger N_1-1-P}B_{o_2}^{\dagger N_2-1+P}|0\rangle \tag{7.20}$$

for $P=(0,\pm 1)$. This calculation is done along a line similar to the one we have used when we only had one type of cobosons, namely, we isolate the norm of states such as $B_{o_1}^{\dagger N_1'}B_{o_2}^{\dagger N_2'}|0\rangle$ with (N_1',N_2') decreasing from (N_1-1-P, N_2-1+P). To do it, we first use Eq. (3.14) to rewrite $B_{o_1}^{N_1}B_m^\dagger$. This leads to four terms. We then use Eq. (3.14) to get $\langle 0|B_{o_2}^{N_2}B_m$ and Eq. (3.15) to get $\langle 0|B_{o_2}^{N_2}D_{om}$. This allows to split $S^{(P)}$ as

$$S^{(P)} = N_1 T_1^{(P)} + N_2 T_2^{(P)} - N_1(N_1-1)U_{11}^{(P)} - N_2(N_2-1)U_{22}^{(P)}$$
$$- N_1 N_2 U_{12}^{(P)}, \tag{7.21}$$

where the contributions $T_1^{(P)}$ and $T_2^{(P)}$ are somewhat direct since they read

$$T_1^{(P)} = \delta_{o_1 m}\langle 0|B_{o_2}^{N_2}B_{o_1}^{N_1-1}|\psi_{n,P}\rangle, \tag{7.22}$$

$$T_2^{(P)} = \delta_{o_2 m}\langle 0|B_{o_1}^{N_1}B_{o_2}^{N_2-1}|\psi_{n,P}\rangle, \tag{7.23}$$

where we gave set $|\psi_{n,P}\rangle = B_n^\dagger B_{o_1}^{\dagger N_1-1-P}B_{o_2}^{\dagger N_2-1+P}|0\rangle$.

The three other terms $U_{11}^{(P)}$, $U_{22}^{(P)}$, and $U_{12}^{(P)}$ contain one Pauli scattering explicitly which describes the fermion exchanges between two cobosons in state o_1, two cobosons in state o_2, and one coboson in state o_1 with one coboson in state o_2, as understood from their N prefactors. These $U^{(P)}$ terms precisely are

$$U_{11}^{(P)} = \sum_i \lambda\begin{pmatrix} o_1 & i \\ o_1 & m \end{pmatrix}\langle 0|B_{o_1}^{N_1-2}B_{o_2}^{N_2}B_i|\psi_{n,P}\rangle, \tag{7.24}$$

$$U_{22}^{(P)} = \sum_i \lambda\begin{pmatrix} o_2 & i \\ o_2 & m \end{pmatrix}\langle 0|B_{o_2}^{N_2-2}B_{o_1}^{N_1}B_i|\psi_{n,P}\rangle, \tag{7.25}$$

$$U_{12}^{(P)} = \sum_i \left[\lambda\begin{pmatrix} o_2 & i \\ o_1 & m \end{pmatrix} + \lambda\begin{pmatrix} o_1 & i \\ o_2 & m \end{pmatrix}\right]\langle 0|B_{o_2}^{N_2-1}B_{o_1}^{N_1-1}B_i|\psi_{n,P}\rangle. \tag{7.26}$$

Their leading contributions in fermion exchanges are obtained by passing B_i over $B_n^\dagger$ in $|\psi_{n,P}\rangle$ through commutator (3.6). The trivial term corresponds to taking $i=n$, while the two other terms generate additional exchanges between i, or n, and the other cobosons in states o_1 and o_2. In the case of $U_{11}^{(P)}$, the remaining matrix element is just $(N_1-2)!N_2!G_{N_1-2,N_2}$ for $P=1$, while it contains additional Pauli scatterings for $P\neq 1$. In the same way, the remaining matrix element of $U_{12}^{(P)}$ is just $(N_1-1)!(N_2-1)!G_{N_1-1,N_2-1}$ for $P=0$, while for other P's, this matrix element contains additional Pauli scatterings. Consequently, the contributions to the $U^{(P)}$'s with only one Pauli scattering reduce to

$$U_{11}^{(1)} \simeq \lambda\begin{pmatrix} o_1 & n \\ o_1 & m \end{pmatrix}(N_1-2)!N_2!G_{N_1-2,N_2},$$

$$U_{12}^{(0)} \simeq \left[\lambda\begin{pmatrix} o_2 & n \\ o_1 & m \end{pmatrix} + \lambda\begin{pmatrix} o_1 & n \\ o_2 & m \end{pmatrix}\right]$$
$$\times (N_1-1)!(N_2-1)!G_{N_1-1,N_2-1}, \tag{7.27}$$

and similarly, $U_{22}^{(1)}$ obtained from $U_{11}^{(1)}$ by changing 1 into 2.

Let us now turn to the $T_1^{(P)}$ term defined in Eq. (7.22). For $P=1$, we pass $B_n^\dagger$ over $B_{o_1}^{N_1-1}$ using many-body commutator (3.14). The contribution without exchange, which is the dominant one at small density, reads

$$T_1^{(1)} \simeq (N_1-1)\delta_{o_1 m}\delta_{o_1 n}(N_1-2)!N_2!G_{N_1-2N_2}. \tag{7.28}$$

If for $P=0$, we do the same but with $B_{o_2}^{N_2}$, we find

$$T_1^{(0)} \simeq N_2 \delta_{o_1 m}\delta_{o_2 n}(N_1-1)!(N_2-1)!G_{N_1-1N_2-1}. \tag{7.29}$$

On the opposite, for $P=-1$, additional exchange processes are necessary to transform the $B_{o_1}^{\dagger N_1}$ operator on the right into $B_{o_1}^{\dagger N_1-1}$ in order to get rid of the $B_{o_1}^{N_1-1}$ operator on the left, as necessary to generate a G_{N_1-1,N_2} factor. By calculating $T_2^{(P)}$ along the same line and by inserting all these matrix elements into Δ_{11} and Δ_{12} given in Eqs. (7.14) and (7.15), we end with the Hamiltonian mean value written in Eq. (7.18).

VIII. DISCUSSION

In Secs. IV–VII, we have performed detailed calculations of the Hamiltonian mean value for three different types of states, namely, a pure state $|\phi_0\rangle = B_o^{\dagger N}|0\rangle$, a coherent state

$|\phi\rangle = (a' B_{o'}^\dagger + a'' B_{o''}^\dagger)^N |0\rangle$, and a fragmented state $|\phi_{12}\rangle = B_{o_1}^{\dagger N_1} B_{o_2}^{\dagger N_2} |0\rangle$ with $N_1 + N_2 = N$. We have taken $N = 2$ first since the calculations are rather trivial and most of the physics can already be understood from this two-body problem. As explicitly shown, the results for large N can be deduced from the ones for $N = 2$ by putting N in front of the scatterings, in this way producing a density correction in $\eta = N(a_B/L)^d$ to the bare energy, as physically reasonable.

We have considered elementary bosons as well as composite bosons. The calculations with composite bosons are done using the new many-body theory designed for them, in which the fermion exchanges appear explicitly through a set of Pauli scatterings $\lambda\binom{n\ j}{m\ i}$. These exchanges are visualized by using Shiva diagrams, which are of great help in driving the algebra in the right direction in case of many-body effects—Sec. VII being quite convincing with respect to the utility of these diagrams. The Hamiltonian used when the composite nature of the particles is retained is the microscopic Hamiltonian for fermions. By contrast, an effective Hamiltonian is necessary when the composite bosons are replaced by elementary particles. In order to recover the exact composite-boson results from this effective Hamiltonian, for the simple problem in which only one interaction scattering appears—the Hamiltonian mean value being first order in the interaction by construction—we must adjust the diagonal scattering of this effective Hamiltonian to be such that

$$\bar{\xi}_{ij;ij} \equiv \hat{\xi}_{ij;ij}, \qquad (8.1)$$

where $\hat{\xi}_{ij;ij}$ is the symmetrical combination of direct and exchange-interaction scatterings of two composite bosons, defined as

$$\hat{\xi}_{ij;ij} = \frac{1}{2}\left[\xi\binom{j\ j}{i\ i} + \xi\binom{i\ j}{j\ i} \right] - \frac{1}{2}\left[\xi^{in}\binom{j\ j}{i\ i} + \xi^{in}\binom{i\ j}{j\ i} \right]. \qquad (8.2)$$

The term $\xi\binom{n\ j}{m\ i}$ corresponds to the interaction scattering of the composite-boson many-body theory which describes fermion interactions without fermion exchange, while the term $\xi^{in}\binom{n\ j}{m\ i} = \Sigma \lambda\binom{n\ q}{m\ p}\xi\binom{q\ j}{p\ i}$ corresponds to the exchange-interaction scattering, with $\lambda\binom{n\ q}{m\ p}$ describing fermion exchange in the absence of fermion interaction. Note that in the case of diagonal scattering, we do have $\xi^{in}\binom{j\ j}{i\ i} = \xi^{out}\binom{j\ j}{i\ i}$, as physically required by the time reversal of the scattering.

Within this identification of the diagonal effective scattering for bosonized particles, we find that the Hamiltonian mean values read the same for elementary and composite bosons in the three states of interest, namely,

$$\frac{\langle \phi_0 | H | \phi_0 \rangle}{\langle \phi_0 | \phi_0 \rangle} \simeq NE_0 + \frac{N(N-1)}{2}\hat{\xi}_{oooo}, \qquad (8.3)$$

$$\frac{\langle \phi | H | \phi \rangle}{\langle \phi | \phi \rangle} \simeq NE_0 + \frac{N(N-1)}{2}(1 + 2|a'a''|^2)\hat{\xi}_{oooo}, \qquad (8.4)$$

$$\frac{\langle \phi_{12} | H | \phi_{12} \rangle}{\langle \phi_{12} | \phi_{12} \rangle} \simeq NE_0 + \left(\frac{N(N-1)}{2} + N_1 N_2 \right)\hat{\xi}_{oooo}. \qquad (8.5)$$

The above results are exact for elementary bosons and only approximate for composite bosons, being valid to lowest order in density only: indeed, in the case of elementary bosons, the many-body physics is induced by 2×2 interactions, while for composite bosons, a quite subtle new set of many-body effects arise from fermion exchanges which can exist between more than two cobosons.

The single-state case is recovered for a' or $a'' = 0$ in Eq. (8.4) and for N_1 or $N_2 = 0$ in Eq. (8.5), as expected. We also see that, since $\hat{\xi}_{oooo}$ must be positive (otherwise the system would suffer a density collapse, its energy decreasing with increasing density), the minimum energy is obtained for macroscopic occupation of the single state $|\phi_0\rangle$. Consequently, a condensate in just one state is stable—a condensate fragmented into two eigenstates or in a superposition of different eigenstates has a macroscopically higher energy, even when the other states (o_1, o_2) or (o', o'') are infinitesimally close to the ground state o.

We end with one last comment. We have here considered the possibility of the condensate to differ from a macroscopically occupied single state through the study of the Hamiltonian mean value for the states $|\phi_0\rangle$, $|\phi\rangle$, and $|\phi_{12}\rangle$. Of course, $B_o^{\dagger N}|0\rangle$ is not the true ground state of the system. It however corresponds to this ground state to lowest order in the interaction, as can be seen from the fact that the Hamiltonian mean value of this state has a zero-order term in scattering which is the expected one, namely, NE_o [see Eq. (8.3)]. It is important to stress, however, that while "lowest order in the interaction" is a well-defined concept in the case of elementary particles, for which the Hamiltonian reads $\bar{H}_0 + \bar{V}$, such a concept has no clean meaning for composite bosons since it is not possible to describe the interaction between cobosons as a potential, and thus define a zero-order Hamiltonian or a zero-order eigenstate.

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APPENDIX: NOZIÈRES' ORIGINAL ARGUMENT

For completeness, we here reproduce the original argument of Nozières[14] on Bose-Einstein condensate, that there are a macroscopic number of elementary bosons in a single quantum state, not from a difference in the free-particle kinetic energy but the exchange part of the interaction energy. Nozières[14] calculates this interaction energy, within the Born approximation, namely,

$$\langle \bar{V} \rangle = \frac{\langle \bar{\phi} | \bar{V} | \bar{\phi} \rangle}{\langle \bar{\phi} | \bar{\phi} \rangle}, \qquad (A1)$$

for a structureless interaction Hamiltonian

$$\bar{V} = \frac{V_0}{2} \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} \bar{B}^\dagger_{\vec{k}_1} \bar{B}^\dagger_{\vec{k}_2} \bar{B}_{\vec{k}_3} \bar{B}_{\vec{k}_1 + \vec{k}_2 - \vec{k}_3}.$$

He first considers an elementary-boson condensate made of a single state $|\bar{\phi}_0\rangle = \bar{B}^{\dagger N}_0 |0\rangle$, the elementary-boson operator $\bar{B}_{\vec{k}}$ being such that $[\bar{B}_{\vec{k}}, \bar{B}_{\vec{k}'}] = \delta_{\vec{k}, \vec{k}'}$. The interaction energy is then found to be, in the large-N limit,

$$\langle \bar{V} \rangle_0 \simeq \frac{1}{2} V_0 N^2. \tag{A2}$$

Nozières then considers a condensate made, not of a single quantum state, but of *two* degenerate or nearly degenerate states, $|\bar{\phi}_{12}\rangle = \bar{B}^{\dagger N_1}_1 \bar{B}^{\dagger N_2}_2 |0\rangle$, with the same total number of particles $N_1 + N_2 = N$. In this fragmented state, the interaction energy is found to be

$$\langle \bar{V} \rangle_{12} = \langle \bar{V} \rangle_0 + V_0 N_1 N_2. \tag{A3}$$

For V_0 positive, as necessary to prevent a density collapse, this readily shows that it we must pay a *macroscopic* amount of energy to break up the condensate into two parts, due to

exchange between these two parts, as seen from the $N_1 N_2$ prefactor of this additional energy.

Instead of the structureless constant scattering V_0 used by Nozières,[14] we have, in this paper, decided to use a scattering $\bar{\xi}\binom{n\ j}{m\ i}$, which *a priori* depends on the in and out states. The idea is to make an easier comparison with the results obtained for composite bosons which read in terms of the specific combination of direct and exchange processes given in Eq. (3.12), namely, $\hat{\xi}\binom{n\ j}{m\ i} = \xi\binom{n\ j}{m\ i} - \xi^{\text{in}}\binom{n\ j}{m\ i}$. This, in particular, allows us to identify the proper effective scattering we must use for bosonized particles. Let us however stress that this identification can only be done for energy-conserving processes as in the case of *diagonal* processes, i.e., processes in which the in and out states are identical. Indeed, such a scattering $\hat{\xi}\binom{n\ j}{m\ i}$ cannot be used in general because, as $[\xi^{\text{in}}\binom{n\ j}{m\ i}]^* = \xi^{\text{out}}\binom{i\ m}{j\ n}$, the resulting Hamiltonian would not be Hermitian, due to Eq. (3.11), which tells that the in and out scatterings are equal only when energy is conserved. This difficulty is basically linked to the fact that there is no way to have an effective Hamiltonian for bosonized particles which is Hermitian *and* valid for all many-body effects. By taking a real constant V_0, Nozières[14] hides this difficulty.

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